AN INTRODUCTION TO FLUID MOTION

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BY

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PREFACE

On adding this small volume to the ever-growing list of what Charles Lamb termed "books which are no books—biblia abiblia," it may be advisable to follow the common custom of explaining on what grounds the addition may be justifiable and useful.

Though Fluid Motion has been studied for many centuries, notable advances both in the understanding of the subject and in its applications have been made in recent years. The improvements in the transmission and metering of fluids, the development of the water and steam turbines, the study of ship and projectile resistance, and the application of the study of Fluid Motion to aircraft, meteorology and methods of power transmission, coming at a time when electrical science has largely occupied the public mind, have been taken for granted and passed almost unnoticed.

This subject, which has hitherto been studied as a branch of mathematics—hydrodynamics—and as hydraulics, a branch of engineering, has clearly reached the stage when it should be recognised in its proper sphere, as an independent subject, forming a branch of physics. Nevertheless, it is generally omitted entirely from college courses in physics, and the corresponding literature is mostly confined to that contained in the publications of scientific societies.

It is the aim of the present work to help to fill this gap in the literature. The book is therefore neither to be considered primarily as an examination textbook nor as a manual for

experimental work, but as an introduction and a guide to one embarking on the advanced study of Fluid Motion. It would be inadvisable, and indeed impossible, to give a complete treatment of the present state of the subject in any one volume. Many references have however been given to the more important original publications, as well as to standard works on the purely mathematical and engineering aspects of the subject. It is assumed that the reader has a knowledge of portions of physics of about pass-degree standard, and of the corresponding mathematics required. The various properties of the fluid are considered in turn, the simpler cases, where few properties are concerned, being treated first. The effect of the other variables is considered later, free use being made of the Method of Dimensions. Experimental work is described throughout the book, but the text is not burdened with too many details of laboratory technique. The study of Fluid Motion is as yet but partially developed, leaving many openings for original work. If this book should help or encourage its study, the work will not have been in vain.

W. N. B.

April, 1925.

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CHAPTER I

OUTLINE OF THE GENERAL PRINCIPLES

Properties of the Fluid.—The motion of fluids is affected by a considerable number of properties of the fluid. In general account has to be taken of density, viscosity, compressibility, surface tension (and in some cases also of the mean free path and effective size of the molecules, when these are comparable with the distance between the boundary walls of the system). The influence of temperature and pressure on the above quantities has also to be considered. But in many cases some at least of the above properties are not involved, or only produce slight effects. The mathematical treatment * of the subject can be considerably simplified by assuming the fluid to have zero compressibility and viscosity. Another method of simplification consists in supposing that the motion is 'irrotational,' so that any small portion of the fluid at a point has no angular velocity about its centre of gravity. Yet another method consists in supposing the velocity at all points small, so that, for a fluid of finite viscosity, the forces due to viscosity (which are proportional to the velocity) are much more important than the forces accelerating the fluid (which are proportional to the square of the velocity).

Compressibility.—The effect of compressibility of the fluid, or its reciprocal, the volume elasticity (defined as

* Lamb, *Hydrodynamics*. Basset, *Hydrodynamics*.

$k = -v\frac{\delta p}{\delta v}$, an increase in pressure divided by the corresponding fractional decrease in volume) is that the density of the fluid changes with the pressure. In the case of liquids this effect is only of importance when compressional waves occur. In the case of gases it is also of importance when the pressure changes by an appreciable fraction, on going from one place to another in the fluid. This occurs when the velocity of the gas is appreciable compared with the velocity of a compressional or sound wave in the gas (see Chapter V). *Whenever surface tension, compressibility and the mean free path of the molecules need not be considered, the flow of a gas corresponds exactly to the flow of liquid through the same system.*

Viscosity.—The viscosity of a fluid may be defined by considering two parallel plane solid surfaces a distance x apart, the space between which is filled with the fluid. When one surface is kept fixed, and the other moved with a constant small velocity u in its own plane, a tangential force F opposing the relative motion is exerted on an area S of each surface, of amount $F = \mu.S.\frac{u}{x}$, where μ is called the coefficient of viscosity of the fluid. This quantity is found to be a constant for any particular fluid, varying considerably with the temperature, but little with the pressure (see Table I). The velocity of the fluid is found to vary directly as the distance from the fixed surface. It is evident that the tangential force per unit area, opposing relative motion, is exerted between any two contiguous layers of the fluid parallel to the surfaces. The force may be written more generally $F = \mu.S.\frac{du}{dx}$, and it is then unnecessary to consider specially the solid boundaries.

A supply of energy is required to maintain the relative

motion, and this energy results in a slight heating of the fluid. This dissipation of energy into heat is supposed to be largely due to the migration of molecules from any one place to another where the velocity is different. The migration is due to the thermal agitation of the molecules (the molecular velocity at random in all directions). In the case of gases, the coefficient of viscosity increases with rise in temperature, as the molecular velocity and mean free path increase. But the forces between the molecules affect the question; and in the case of liquids these forces become specially important, and the viscosity decreases with rise in temperature. The following table of viscosities is compiled from some of the more accurate determinations.

TABLE I

Fluid.	Values of the viscosity, μ, in C.G.S. units.							
	0° C.	10° C.	20° C.	30° C.	40° C.	60° C.	80° C.	100° C.
Water	·0179	·0131	·0101	·0080	·0066	·0047	·0036	·0028
Mercury . . .	·0169	·0162	·0156	·0151	·0146	·0137	·0129	·0122
Carbon bisulphide	·0043	·0040	·0037	·0034	·0032	—	—	—
Glycerine * . .	46·0	21·0	8·5	3·5	—	—	—	—
	Values of $\mu.10^6$							
Air	170	175	180	185	189	199	208	217
Hydrogen . . .	86	88	90	92	95	99	103	107
Nitrogen . . .	166	171	176	181	185	194	203	212
Carbon dioxide .	137	142	146	151	156	166	175	184

* The above values refer to what is usually supplied under the name of glycerine. Pure glycerine melts at about 17° C.

Solid Boundaries.—It is found that *fluid at any solid boundary walls is almost invariably at rest relative to these walls.* This effect causes experimental results to differ from those calculated on the assumption that the viscosity may

be considered zero. At places near to any solid wall, the fluid moves parallel to the wall with a relative velocity that is almost proportional to the distance from the wall, elements of the fluid have an angular velocity about their centres of gravity, and the motion cannot therefore be assumed 'irrotational.'

Velocity of the Fluid.—The direction of motion at a point in a fluid may be found by photographing or observing the figures formed by introducing dust, small air bubbles, or streams of coloured liquid (in a liquid), or smoke or dust (in a gas). The velocity may be found by measuring the distance dust or other particles move in some short known time.* The velocity may also be measured in some cases by means of a Pitot tube (see equation 1·06). The cooling produced in a metal wire, through which an electric current is passing, by the movement of the surrounding gas may also be used to estimate the velocity of the gas.† (And two such wires may be used to find the direction of motion also.)

Stream Lines, Eddies and Turbulent Motion.—By the above means two more or less distinct kinds of motion are found to occur. In the one form the motion is said to be 'stream line motion.' Every particle passing a chosen point follows the same path or 'stream line' (unless the velocity of the whole fluid be varying with time). This form of motion is illustrated in Fig. 1, which shows the stream lines round a small obstacle placed between two parallel sheets of glass between which glycerine was passing (Hele Shaw).‡ Colouring matter was introduced to detect the

* *Dict. of Applied Physics*, Vol. v, Model Experiments in Aeronautics, p. 191.

† Morris, *Engineering*, p. 178, 1913. King, *Phil. Mag.*, p. 556, 1916. Thomas, *Proc. Phys. Soc.*, xxxii, p. 196, and p. 291 (1920). Davis, *Phil. Mag.*, p. 1057, 1924.

‡ [illegible] *Naval Architects*, 1898, p. 27.

stream lines. In such cases heat is being liberated owing to the viscous forces between adjacent stream lines. This dissipation of energy into heat, largely due to the migration of molecules to places where the velocity is different, may take place not only in directions at right angles to the stream lines, but also in the direction of the stream lines, if the lines converge or diverge and the velocity varies along the same lines. *If the motion be slow enough, the size of the bodies concerned small enough, or the viscosity large enough, stream line motion can always be obtained.*

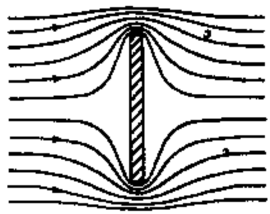


FIG. 1.

One or more eddies or vortices may occur, in which the liquid exhibits continuous rotation. Particles of the fluid may thus circulate round closed paths in the eddy an indefinite number of times. This kind of motion is shown in parts of Fig. 2 (Nisi and Porter),* obtained by photographing filaments of smoke, etc., round an obstacle placed in a channel through which air was flowing. Transverse sections of the eddies are seen in this diagram. The 'moment of momentum' of an eddy can only be changed by the action of tangential forces, due to viscosity, acting at its boundary; and hence an eddy could never be produced nor decay in a fluid of zero viscosity. Also it can be shown (p. 70) that the long filament of rotating fluid forming the eddy must either form a closed ring (as in the case of a smoke

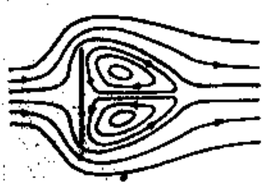


FIG. 2.

* *Phil. Mag.*, Vol. xlvi, plate xii, Nov. 1923.

ring), or must end on the boundary walls of the system.

Eddies may be stationary in position relative to the solid walls, as in Fig. 2; or they may move along in the main stream of fluid, their shape changing. When such motion of the eddies occurs, the magnitude and direction of the velocity at any chosen fixed point vary periodically, in a more or less irregular manner. Turbulent motion is then said to occur. As in the case of stream line motion, energy is being dissipated in the form of heat. The rise in temperature produced is generally unimportant; but the corresponding loss of kinetic and other forms of energy in the stream cannot be neglected. The turbulent motion occurring when fluid flows in a straight tube, will be considered in Chapter II; and in Chapter IV cases of turbulent motion behind a body that is moving through fluid will be discussed.

Equations of Motion.—Mathematical equations of motion* have been derived, relating the pressure gradient in any direction with the velocities at neighbouring points, and the density and viscosity of the fluid. An equation may be deduced from the fact that the mass of fluid entering any region is equal to the mass leaving in the same time, unless the density at the place is changing with time. This equation is known as the equation of continuity. Other relationships may be derived when there is no velocity normal to some rigid boundary; or when there is also no tangential velocity relative to the boundary. These equations are known as the boundary conditions. The equations are complicated, and have only been solved in certain classes of cases. We shall, therefore, not attempt the problems by these general means, but consider the simplest cases, and proceed to obtain as general an understanding as possible of the physical aspect of the problems.

* Lamb, *Hydrodynamics*. Basset, *Hydrodynamics*. Stokes, *Camb. Phil. Trans.* (1845), viii, or *Math. and Phys. Papers*, i. 80.

Bernoulli's Theorem.—Let it be assumed that, in a case of steady stream line flow, the viscous forces can be neglected as a first approximation. Consider a 'Tube of Flow' bounded by stream lines (Fig. 3). Let the fluid have constant density ρ; let a_1, a_2 be areas of transverse section of the tube; u_1, u_2 velocities at these sections; h_1, h_2 the heights of these places above any arbitrary zero of level; and p_1, p_2 the pressures at the two places.

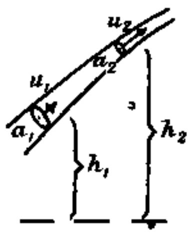


FIG. 3.

The mass of fluid m passing across any transverse section per second, is then given by

$$m = a_1 u_1 \rho = a_2 u_2 \rho \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}01)$$

or $\quad a_1 u_1 = a_2 u_2 =$ constant, along the tube.

This statement, which is a consequence of the fact that equal masses enter and leave the portion considered per second, is a special case of the 'equation of continuity.'

When unit mass passes across each section, the amount of work done by the forces exerted by the adjoining fluid is obtained by multiplying the pressures by the volumes swept out, and subtracting: $p_1\left(\frac{1}{\rho}\right) - p_2\left(\frac{1}{\rho}\right)$.

The corresponding gain in potential energy is $(h_2 - h_1)g$. And the gain in kinetic energy, when the velocity at each point does not vary with time, is equal to the difference between that of the liquid leaving and entering the section in the time, $\frac{1}{2}(u_2^2 - u_1^2)$.

Hence, if there be no frictional forces, we have

$$\frac{p_1 - p_2}{\rho} = (h_2 - h_1)g + \tfrac{1}{2}(u_2^2 - u_1^2)$$

$$\text{or} \quad \left.\begin{aligned} \tfrac{1}{2}\rho u_1^2 + h_1 g\rho + p_1 &= \tfrac{1}{2}\rho u_2^2 + h_2 g\rho + p_2 \\ &= \text{constant, along the tube.} \end{aligned}\right\} \quad (1{\cdot}02)$$

This result, based on the conservation of energy when there are no viscous forces, is known as Bernoulli's equation (and was first stated by D. Bernoulli in 1738). Since the first and second terms in the equation represent the kinetic and potential energies of unit volume of the fluid respectively, the third term is sometimes described as the 'pressure energy per unit volume' at the point. It must not be supposed that this term is equal to the work done in compressing unit volume of the fluid. The latter is equal to $p^2/2k$ where k is the volume elasticity. (In the present case as ρ is supposed constant, k is infinitely large and no work is done in compressing the fluid.) For steady, frictionless irrotational motion equation (1·02) can be proved constant throughout the fluid.

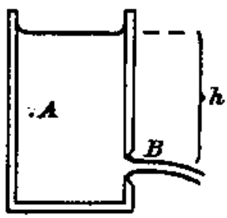


FIG. 4.

Flow through an Orifice.—Let us apply this equation to a simple experiment—the flow of liquid as a jet from a small hole or orifice in the side of a tank of liquid. (The level of the liquid surface in the tank is supposed to change only slowly. Or liquid may be poured in to keep the level up to a certain overflow hole.)

Consider a stream line from A to B (Fig. 4), a point on the surface of the jet, choosing the zero of level through B. If the pressure of the air on the liquid surface in the reservoir be denoted by P, the sum of the pressure and potential energy per unit volume at A (where the velocity is very small) is $P + hg\rho$. At B, the pressure is equal to that of the surrounding air, $P + hg\rho'$ (where ρ' denotes the density of the air), if the effect of surface tension can be neglected. The terms in the equation then become $\frac{1}{2}\rho u^2 + (P + hg\rho')$. Equating the expressions found for the points A and B we

obtain $\frac{1}{2}\rho u^2 = hg(\rho - \rho')$, or neglecting the pressure difference due to the column of air of height h we have

$$\tfrac{1}{2}\rho u^2 = hg\rho \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}03)$$

The calculated velocity is equal to that attained by a body falling freely the vertical distance h; and it varies slightly from the top to the bottom of the jet.

The liquid just before emerging has a component velocity radially inwards towards the axis of the jet. The jet is thus tapering near the orifice. There is an outward acceleration of the liquid and a corresponding variation in pressure and velocity across the transverse section. At a short distance from the orifice, however, the jet is found to become approximately cylindrical (the 'vena contracta'). Here the stream lines are almost parallel, and there is no radial acceleration of the liquid. The whole of the jet is subject to the downward acceleration due to gravity, and the pressure is the same (atmospheric) at all parts of the jet. Thus equation (1·03) applies to all parts of the jet at the 'vena contracta,' the velocity being slightly greater towards the lower side of the jet.

If the jet be arranged to flow vertically upwards, it is found to rise a height equal to about 0·94.h. The 6 per cent. difference in level between the tip of the jet and the liquid surface in the reservoir represents the portion of the energy converted into heat by viscous forces. The velocity is thus about 3 per cent. less than that calculated by the equation. This velocity may also be determined by letting the jet emerge horizontally and measuring the shape of the parabolic curve it describes under the action of gravity.

Next suppose the jet to emerge from the bottom of the tank. Let a_1 be the area of section of the jet at one point, and a_2 the smaller area of section a distance h lower.

Applying equation (1·02) to the jet we obtain

$$\tfrac{1}{2}\rho(u_2^2 - u_1^2) = hg\rho$$

since the pressure is everywhere approximately atmospheric.

By equation (1·01) this becomes

$$\tfrac{1}{2}\left(\frac{m}{\rho}\right)^2\left(\frac{1}{a_2^2} - \frac{1}{a_1^2}\right) = hg.$$

By measuring all the quantities concerned, the extent of the effect of viscosity, et cetera, can be estimated. The effect is found to be very small. It is probably chiefly due to the force between the air and the jet surface. (The jet has here been assumed approximately cylindrical at any one place.)

It is evident that equation (1·02) cannot be applied to the case of a jet falling into a reservoir of liquid. For all the kinetic energy is converted into heat. If the motion be studied by means of coloured filaments of fluid (injected into the upper reservoir) it is found that eddies are produced where the jet falls into the lower reservoir, but that the flow through the orifice and in the jet is in stream lines. Thus, large production of eddy motion at any place causes equation (1·02) to become quite inapplicable in that region. The heating produced by water falling a considerable distance into a pool was used by Joule as one means of estimating the mechanical equivalent of heat. A similar conversion of all the kinetic energy into heat occurs when liquid flows from a tube into a reservoir of liquid, the edge where the tube joins the reservoir wall not being rounded off at all.

The mass of liquid passing per second through a circular orifice cannot be simply calculated by considering the velocities at points in the plane of the orifice. But by considering the section where the jet becomes cylindrical, the mass per second can be estimated with fair accuracy. Let the area of section at the cylindrical part be given by $C_c\pi r^2$ where r is the radius of the orifice, and C_c the ratio of the areas or

'coefficient of contraction.' (It is found that C_c has a value of about 0·63 in most experiments.) Denote the actual velocity by $C_v u = C_v\sqrt{2gh}$ where C_v is the 'coefficient of velocity' ($C_v = 0{\cdot}97$ approximately in many experiments). Then the mass passing per second, m, is given by

$$m = (\pi r^2 C_c)(C_v \cdot \sqrt{2gh}) \cdot \rho \quad . \quad . \quad . \quad . \quad (1{\cdot}04)$$

The product $C_c C_v = C$ (the 'coefficient of discharge') can be estimated both by measuring the area of section and the velocity, and also by measuring the mass per second m. The results are in as good agreement as can be expected from the method of developing the theory.

Dimensional Treatment of Flow through an Orifice.—Let us next consider the matter from a more general point of view. The mass passing per second through the orifice depends on the difference between the pressure of the air round the jet and the pressure at a point in the reservoir on the same level, where the velocity is negligible. This pressure difference may be written $p_{12} = hg\rho$; and the rate of flow would be the same if p_{12} were unchanged, even if the pressure were not due to the height of liquid in the reservoir, but were produced by means of a pump. Hence, neglecting the effect of viscosity and surface tension, we have to find a relationship between m and r, p_{12} and ρ. Let this be written

$$m = \phi(r, p_{12}, \rho).$$

The expression on the right-hand side must have the dimensions of mass per second. (It is meaningless to equate m to any other quantity than mass per second. This may be understood by supposing the unit in which mass or time is measured to be changed. If the two sides of the equation had different dimensions when expressed in terms of mass, length and time, such a change of units would multiply the two sides of the equation by different quantities, and the

equation could not be true for both the original and the final systems of units.)

The dimensions of the variables are :—

m	MT^{-1}
r	L
p_{12}	$ML^{-1}T^{-2}$
ρ	ML^{-3}

The only expression involving r, p_{12} and ρ that has the same dimensions as m is $A.r^2.\sqrt{p_{12}.\rho}$ where A is a numerical constant (having zero dimensions).

Hence we have

$$m = Ar^2.\sqrt{p_{12}.\rho} = Ar^2.\sqrt{hg\rho^2} \qquad (1{\cdot}05)$$

(In equation (1·04), A was seen to have the value $C_cC_v\pi\sqrt{2}$.)

If a square orifice had been used, r being used to represent the length of a side of the square, the dimensional proof would have been the same, but A in equation (1·05) would have had a different value. A similar argument applies to an orifice of any other specified shape. Any change in the ratio of the thickness of the wall in which the orifice was made to the orifice diameter; or change of the ratio of depth to width of the orifice, might, of course, cause a change in the value of A. It has been assumed in the dimensional treatment that a single length (r) is sufficient to specify the orifice. Hence, for A to be unchanged, when r is changed, the orifice must be made of different size but of the same geometrical proportions.

If an orifice be situated in a wall separating two reservoirs of liquid, the rate of flow can be represented by equation (1·05), where h now represents the difference between the levels of the liquid surface in the two reservoirs. It is found that A has about the same value as it had when the liquid emerged into air in the form of a jet. Hence we conclude that surface

tension in the latter case is not important; and also that when liquid emerges into the second reservoir of liquid, it probably moves in a similar jet, surrounded by liquid in slow motion. (This is found to be the case by investigating the motion with the help of coloured filaments of liquid.)

The Pitot Tube.—Let a solid body be placed in liquid that is originally moving in straight parallel stream lines, as in Fig. 5. Let u be the velocity of the fluid at a distance from the body, measured relative to the body. The stream lines will be found to divide (as in Figs. 1 and 2) with a result that at B there is no relative velocity of the fluid normal to or parallel to the surface of the solid. Let us consider a very narrow tube of flow along the line AB (where A is a point remote from the body), and compare it with the corresponding tube when the solid body is removed. The conditions at A are not sensibly changed. But in the one case there is no velocity at B, and in the other case, when the body is removed, the velocity at B is the same as at A. Denoting the pressure at B by p_1 and the pressure there when the body is removed (and the velocity is u) by p, we find by equation (1·02) that

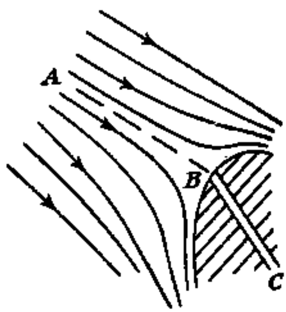


Fig. 5.

$$p_1 + 0 = p + \tfrac{1}{2}\rho u^2$$

or

$$\tfrac{1}{2}\rho u^2 = (p_1 - p) \quad . \quad . \quad . \quad . \quad (1{\cdot}06)$$

Thus, by measuring the excess of the pressure at B over the pressure there when no body is present, the velocity of the fluid at B when the body is not present is found. If the

flow be turbulent at A, Bernoulli's theorem cannot be applied, and the above equation can only be supposed approximately true.

The pressure at B can be found by drilling a small hole in the solid body and connecting the end C to some form of pressure measuring instrument (manometer).* Since the hole BC must end at the point B where the stream lines divide, the outside surface of the solid body may be made a surface of revolution about the axis BC, and BC be arranged parallel to the direction of motion of the fluid. (The form of the tube BC is immaterial; and the outer surface need not necessarily be a surface of revolution, provided the point of zero velocity comes at the end of the hole at B.) This apparatus is known as a Pitot tube.

The Static Pressure Tube.—In many cases it is necessary to make some special arrangement to determine what the pressure p at the point B, or near it, would be were the Pitot tube not present. Let us consider the pressure at the point F in Fig. 6. If the lines of flow at DEF are parallel to the side

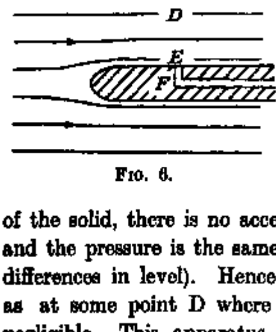


FIG. 6.

of the solid, there is no acceleration in the direction D E F, and the pressure is the same at all these points (apart from differences in level). Hence the pressure at F is the same as at some point D where the effect of the solid body is negligible. This apparatus is known as a static pressure tube. If no flow takes place along the tube F, the pressure is the same at all points in it (apart from the effect of difference of level), and it may be measured by means of a manometer connected to the other end of the hole that is drilled at F. The pressure at the point D which is thus

* See Chapter II.

measured, is not that at the point B (Fig. 5) when no Pitot tube is present, unless the two parts of the apparatus, Pitot tube and static pressure tube, be near together but do not interfere with one another. It is often suitable to use a single bar or solid for the Pitot and static pressure holes.

Velocity Measurement with a Pitot Tube.—Fig. 7 illustrates a well-designed form of Pitot tube, such as is used at the National Physical Laboratory. The central tube, the open end of which is to face the arriving fluid, is surrounded by a concentric tube, in the sides of which small holes are drilled to observe the static pressure p. The outer tube is tapered towards the tip, and the stream, after diverging, flows approximately parallel to the sides of the tube. The side holes are at such a distance from the tip that the stream lines have become sensibly parallel, and the correct static pressure is measured. The Pitot and static pressure holes are connected by tubes to the two sides of a suitable manometer. For accurate measurement of the velocity of a gas, the Chattock tilting manometer (Fig. 20) may be used.

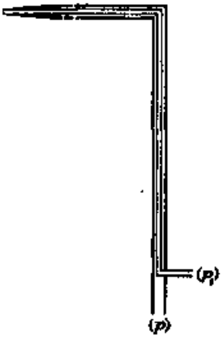


FIG. 7.

Owing to the effects of turbulence, viscosity, and possible faulty design of the tube, it is desirable to test experimentally the accuracy with which equation (1·06) is applicable to the tube. Two methods of experimenting may be used. The tube may be placed in moving fluid, or may be moved through stationary fluid. A careful test of such a tube was carried out at the National Physical Laboratory,* by mounting the

* *Report of the Advis. Committee for Aeronautics*, 1910–11, and 1912–13.

tube in air, at the end of a rotating arm. An allowance was made for the fact that a slow general circulation of the air is set up by the arm, and the velocity of the tube relative to the air is less than its velocity relative to the ground. As a result of the experiments, it was decided that, over the range covered by the measurements, equation (1·06) is applicable to a properly designed tube to an accuracy of 1/10 per cent. Experiments on the deviation from the simple equation, when the effect of viscosity is specially large, are referred to in Chapter II.*

The simplest use of a Pitot tube is to insert it in the jet of liquid emerging from an orifice. The tube must be so small that it does not appreciably affect the flow in the jet. The end, C, of the hole (Fig. 5) is connected to the lower end of a vertical glass tube, that is open at the top. Since the pressure in the jet and at the open end of the glass tube is atmospheric (p), and the pressure in the Pitot tube is p_1, the height h of the liquid column in the tube is given by $(p_1 - p) = hg\rho = \frac{1}{2}\rho u^2$. The liquid column is found to reach to within about 6 per cent. of the liquid in the reservoir, confirming the previous observation that the actual velocity is about 3 per cent. less than that calculated by equation (1·03).

Conservation of Momentum.—If the vessel shown in Fig. 4 be suspended freely by means of wires, it will be found to move in the opposite direction to the emerging jet, when the orifice is opened, and to remain deflected whilst flow is taking place. The total momentum of the liquid emerging in any time is equal to the gain in momentum of the reservoir in the reverse direction in the same time (if it be free to move). In other words, a force acts on the reservoir, in the opposite direction to that in which the liquid emerges, of amount equal to the momentum of the fluid leaving per second. If a_1 be the area of cross section of the cylindrical part of the jet,

* Barker, *Proc. Roy. Soc.*, Vol. 101A, p. 435 (1922).

where the velocity is u, the force acting on the reservoir is

$$(a_1 u\rho)u = a_1 u^2 \rho.$$

Let us next suppose the vessel fixed, and consider the forces acting on the liquid in it and in the jet as far as the section a_1. The forces differ from those acting when the orifice is closed, by the force in the opposite direction to the direction of emergence of the liquid, of amount $a_1 u^2 \rho$; by a decrease of the force due to the wall containing the orifice, of amount $(a.hg\rho)$, because of the area a of the wall being cut away; and by a further decrease in the force exerted by this wall, on account of the fact that the liquid near its surface is in motion and is therefore at a less pressure than when no flow occurs. Denoting this last decrease by f, we have for equilibrium

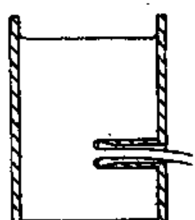


FIG. 8.

$$ahg\rho + f = a_1 u^2 \rho = a_1(2hg\rho)$$

(if viscosity be neglected).

It is difficult to calculate f, as the velocity at points near the surface of the wall containing the orifice is not known. By using a re-entrant tube, however, known as Borda's mouthpiece (Fig. 8), the velocity at this wall can be made very small. Then, apart from the effect of viscosity, we have $ahg\rho = 2a_1 hg\rho$, or the area a_1 of the section of the jet is equal to half that of the orifice a. Hence we have

$$m = a_1 u\rho = \frac{a}{2}\rho.\sqrt{2gh}, \text{ which corresponds to equation (1·04),}$$

the constant C having now the value 0·5. This equation may be verified experimentally, care being taken in measuring the area of the tube a, and also in preventing the jet wetting the inside walls of the tube.

Deflection of a Jet of Fluid.—A force is exerted on a flat plate by a jet impinging on it and leaving it tangentially at its edges (Fig. 9). Since the liquid has finally no velocity normal to the plate, the rate of decrease of momentum in the direction A B is equal to $mu \cos \theta$ (where m is the mass arriving per second and u its velocity at arrival). Thus there is a force acting on the plate in the direction A B equal to $mu \cos \theta$. (There may also be a tangential force in the direction B C, due to viscosity.)

The pressure at the point B, where the stream divides, is

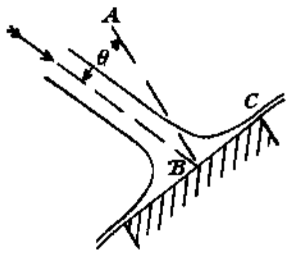


FIG. 9.

in excess of that of the atmosphere by an amount $\frac{1}{2}\rho u^2$. The pressure at C is only slightly larger than atmospheric. Apart from viscosity no work is done by the fluid, if the plate is fixed, and the energy of the fluid after it leaves the plate is the same as at arrival. (The velocity u is strictly that of the jet at B when the plate is removed.) An arrangement, based on this jet impact, has been used to estimate the velocity of an air jet.*

If the jet be deflected as in Fig. 10, there will be a force in the direction A B, of amount $2mu \cos \theta$ (if the difference in level of the entrance and exit be so small that the velocity

* Montel and Foà, Accad. Sci. Torino, *Atti*, 57, 14A, p. 277, 1921–22.

at both places is sensibly equal to that of arrival u). If, in either of these cases, the plate be allowed to move in the direction of the force, work will be done, and the liquid will leave with less energy than it had at arrival. Many machines (turbines and pelton wheels) based on this change

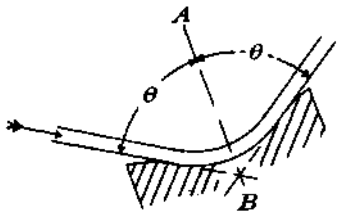


FIG. 10.

of momentum, are described in detail in engineering text-books, and will not be considered further here.*

Curved Stream Lines.—In any curved stream line there is an acceleration towards the centre of curvature. If this be not produced by the action of gravity (as in the case of the parabolic curve described by a jet starting at an angle to the vertical), the pressure must be greater at the outside than at the inside of the curve. Consider circular stream lines about a centre O, Fig. 11. The force acting on a small cylinder of fluid A B, of length δr and area of base S, is $\{(p + \delta p) - p\}S$, towards O. The acceleration towards O is u^2/r, and hence

$$\rho(S.\delta r) \, . \frac{u^2}{r} = S.\delta p$$

or

$$\frac{dp}{dr} = \rho\frac{u^2}{r}. \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}07)$$

This equation may be applied to the case of liquid placed

* See *Dict. of Applied Physics*, Vol. I, Hydraulics, p. 519, also books on Hydraulics by Gibson, Lea and Unwin.

in a vertical cylindrical vessel, open at the top, which is rotated about its axis with a constant angular velocity ω (Fig. 12). (The effect of tangential forces between the air

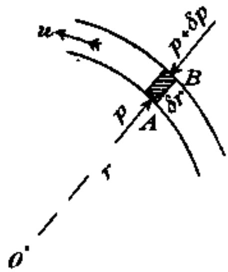


Fig 11.

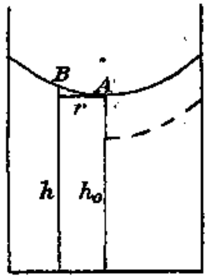


Fig. 12.

above and the liquid surface are supposed negligible, so that the liquid all rotates with the same angular velocity as the vessel.)

At any place in the liquid the pressure increases as we go downwards and as we go radially outwards. Hence, measuring heights from the base upwards, we have

$$\frac{dp}{dh} = -g\rho$$

and
$$\frac{dp}{dr} = \rho\frac{u^2}{r} = \rho\omega^2 . r,$$

or $[p] = \rho\omega^2\left[\frac{r^2}{2}\right]$ for changes in r only.

Since the pressure at A and B is the same, we have

$$(h - h_0)g\rho = \tfrac{1}{2}\rho\omega^2 r^2 \quad . \quad . \quad . \quad . \quad (1{\cdot}08)$$

The curve A B is thus parabolic. The pressure is constant along any one parallel parabolic curve (shown dotted in the diagram).

CHAPTER II

FLUID FLOWING BETWEEN SOLID BOUNDARIES

Pressure Differences; Elimination of the Effect of Difference of Level.—When liquid (or slow gas) flow takes place through tubes of various kinds, the conditions are not much affected by the means of supply and removal of the fluid. No exposed liquid surfaces are directly concerned; and the differences of level previously discussed must be replaced by pressure differences between points in the fluid.

Consider any two chosen points in the system. Denoting the pressures at the points by p_1', p_2' when the fluid is at rest, we have by equation (1·02),

$$(p_1' - p_2') = (h_2 - h_1)g\rho.$$

When the fluid is in motion the pressure difference between the points will, as a rule, be different (say $p_1 - p_2$). If the difference in pressure in the first case be deducted from the value when flow is taking place, we have

$$(p_1 - p_2) - (p_1' - p_2') = (p_1 - p_2) + (h_1 - h_2)g\rho = p_{12} \quad . \quad (2·01)$$

the difference in pressure after deducting that due to difference in level. This quantity is zero when the flow is zero, and usually increases in numerical value as the rate of flow increases. Thus p_{12} and m are related to one another and the remaining variables. Differences in level are not important; and absolute pressure only affects the density of the fluid slightly (since compressibility is supposed unimportant; see Chapter V).

For two points on the same stream line, when the effect

of viscosity can be neglected, equation (1·02) becomes in the new notation

$$p_{12} = \tfrac{1}{2}\rho(u_2^2 - u_1^2) \quad . \quad . \quad . \quad . \quad (2{\cdot}02)$$

Manometers.*—In one of the simplest forms of manometer for measuring a pressure difference, the two points in a liquid, A, B, between which the pressure difference is required, are connected each to a vertical glass tube. (Fig. 13 illustrates

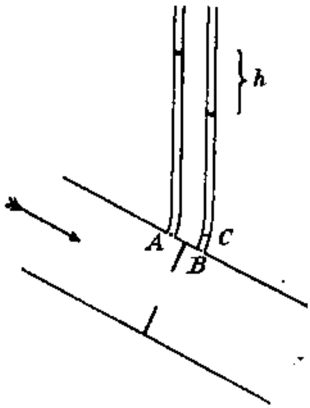


FIG. 13.

this, where the two points are at the walls of a tube, on either side of a diaphragm with an orifice in it.) When no flow is taking place, there will be no difference in level between the liquid surfaces in the two tubes. And when flow is taking place, the difference in pressure indicated, is that between the points A and C; or that between A and B reduced by an amount equal to the difference in pressure between C and B. Thus we have $p_{12} = hg\rho$. The glass tubes should

* Parnell, *The Measurement of Fluid Velocity and Pressure.* J. L. Hodgson, Inst. Marine Engineers, 1924.

be wide to prevent any appreciable difference in level due to the surface tension affecting the two differently. Care must be taken to prevent any air bubbles collecting in the connecting tubes. This may be partly avoided by placing A and B on the *lower* side of the main tube.

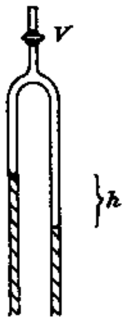


FIG. 14.

To prevent the liquid rising in the manometer to an inconvenient height, or fluctuating in height, the two open tubes may be joined at the top (Fig. 14) and air pumped in through a valve V till the surfaces are at a convenient level. We then have

$$p_{12} = hg(\rho - \rho') \quad . \quad . \quad . \quad . \quad . \quad (2{\cdot}03)$$

where ρ' is the density of the air in the manometer (ρ' is generally negligible compared to ρ). By keeping the volume of the air in the inverted 'U' tube small, the equal change in level of the two liquid columns when the absolute pressure is changed, is made small.

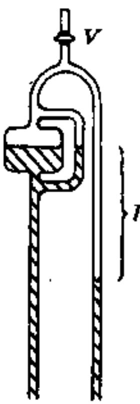


FIG. 15.

A better method is illustrated in Fig. 15. In this case the tube connected to the point of higher pressure is enlarged towards its upper end to form a wide shallow reservoir, in which the one liquid surface is situated. Since the air in the apparatus is mostly in this reservoir, a small rise or fall of the two liquid surfaces by equal amounts would produce an appreciable fractional change in the volume and pressure of the air. Thus any fluctuation of the absolute pressure would have little effect on

the liquid levels. The reservoir level thus keeps almost constant; and it may be read from time to time by means of a short side tube.

An almost complete elimination of the effect of change of absolute pressure may be obtained by using an inverted '∪' tube, the upper part of which is filled with a second liquid of smaller density. The pressure difference is then given by equation (2·03), where ρ' represents the density of the upper liquid. The two liquids must be such as not to mix, and must form a suitable definite meniscus.

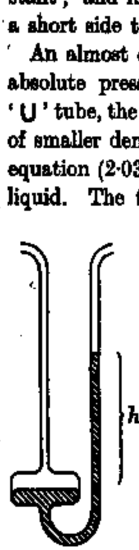


FIG. 16.

For larger pressure differences, or when the main fluid is a gas, a '∪' tube may be used, the lower part containing mercury (or a liquid more dense than the main fluid), the main fluid filling the upper parts and the connecting tubes. If desired the side connected to the larger pressure may be enlarged to form a reservoir near the bottom (Fig. 16). If the area of the liquid surface in the reservoir is large compared with that in the other tube, the surface in the reservoir will change little in level. Equation (2·03) can again be used, where ρ' represents the density of the upper fluid, ρ that of the lower. The liquid surface of separation is of course lower in the limb connected to the point of higher pressure.

The maximum pressure difference that may be measured by manometers of these types is limited chiefly by the length of the liquid column that is convenient. For steady pressures, the difference in level, h, may be measured to 0·01 cm. or less; but for fluctuating pressures the error is considerably larger. For periodically varying pressure differences, the accuracy of measurement of the mean pressure difference may be increased by making part of the manometer or

connecting tubes narrow, so as to decrease the fluctuations of the manometer. But it is necessary in this case to consider what law of damping is introduced, and whether the pressure difference deduced is the arithmetic mean, or some other mean value.*

Manometers for Small Pressure Differences.—Small differences of pressure may be measured by viewing the liquid

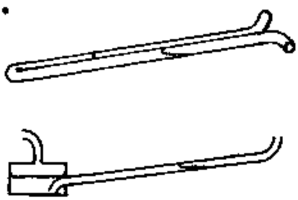


FIG. 17.

surfaces through a microscope; or by inclining the tubes of the manometer as in Fig. 17; or by making the densities ρ and ρ' almost equal. (By using a mixture of two liquids for the manometer liquid it may be arranged to have almost

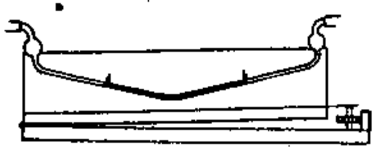


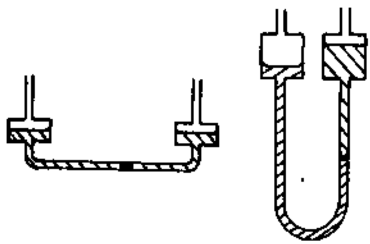
FIG. 18.

the same density as the main liquid. But the increase in sensitivity may be more than compensated by the increase in irregular errors due to surface tension.)

A manometer with inclined tubes (Fig. 18) may be tilted by means of a micrometer screw, till the liquid surfaces are

* Hodgson, *Proc. Inst. Civil Engineers*, Vol. cciv, pp. 134 and 161, 1918.

at the same part of the tubes as when no flow was taking place (and no pressure difference occurred). The difference in level of the two surfaces is then equal to the sine of the angle of tilt, multiplied by the original horizontal distance between the liquid surfaces. The estimation of this distance is subject to some uncertainty, and the manometer cannot in consequence be considered very accurate. Two sensitive forms of differential manometer are shown in Fig. 19. Liquid fills a 'U' tube with enlarged ends and narrow central portion (a). In the central part a small bubble of another liquid or of air is inserted to act as an indicator. Or the apparatus



(a) (b)

FIG. 19.

shown in Fig. 19b may be used; two liquids of almost the same density are placed in the two limbs, and the meniscus of separation is observed. Or two such liquids may be used in the form of apparatus known as a Chattock* manometer (Fig. 20), the whole apparatus being tilted to keep the meniscus separating the liquids at the end of the tube in the central chamber stationary (as viewed by a microscope). With this form of manometer, surface tension errors are minimised and pressure differences up to 2 cm. of water column may be measured to an accuracy of 0·0003 cm. These manometers

* *Phil. Mag.*, p. 450, 1910.

(Figs. 16 to 20) can be adapted for use when the main fluid is liquid or gaseous.

A micrometer screw placed above a mercury surface (contact with which is detected by sending an electric current through a circuit containing the screw, contact, and mercury) may be employed to form a very sensitive manometer. Newton's rings may also be used to detect the relative motion of a fixed and a floating surface. Or two such parallel surfaces may be used to form a condenser, the relative motion being detected by changes in the capacity of the condenser.

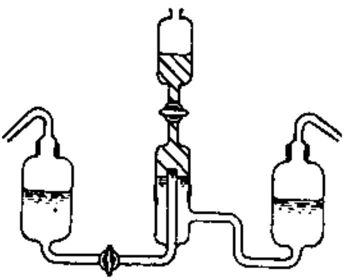


FIG. 20.

Instantaneous pressures may be measured by a moving diaphragm supporting a mirror, or by the piezo-electric charge produced by compressing a crystal, the charge being detected by charging a pair of plates situated in a Cathode Ray tube and photographing the deflection of the rays. A magnetic field, varying periodically with a known frequency, arranged to deflect the Cathode rays in a direction at right-angles, enables time measurements to be obtained.*

Manometers may be made to give graphical records, by photographing the liquid surface, or by actuating a pen.

* Keys, *Phil. Mag.*, p. 473, 1921.

A floating "bell" such as is shown in Fig. 21 is suitable for the latter purpose.

The Venturi Tube.—We will next apply equation (2·02) to an apparatus known as a Venturi tube, used in conjunction with a manometer. This tube devised by Clemens Herschel in 1881 for the measurement of the mass of liquid passing per second (and named in honour of an experimenter in Hydraulics), consists of a converging and subsequently gradually diverging portion inserted in a length of parallel tube (Fig. 22).

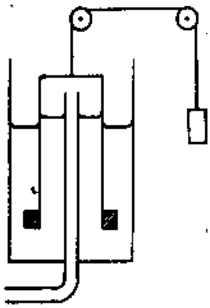


FIG. 21.

By using coloured filaments, it is found that the flow in the entrance or 'upstream' cone is stream line; but that in the exit or 'downstream' cone is often turbulent, unless the angle of the cone is small and the rate of flow slow. It is found, as might be expected from this, that Bernoulli's equation cannot be used to calculate p_3-p_1 accurately; but repre-

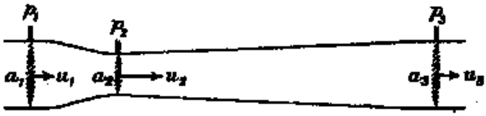


FIG. 22.

sents the relationship between p_1-p_2 and the other variables concerned (i.e. for the entrance cone) fairly accurately. Even though the flow is in stream line in the entrance cone, viscosity will affect the question, and the velocity will not be the same all over the transverse section. (The exit cone is made of small angle to reduce the total energy dissi-

pated into heat in the tube to as small an amount as is practicable.)

By Bernoulli's equation (equ. 1·02 and 2·02) we see that the pressure p_2 is less than p_1 by an amount

$$p_{12} = \tfrac{1}{2}\rho(u_2^2 - u_1^2) = \tfrac{1}{2}\frac{m^2}{\rho}\left(\frac{1}{a_2^2} - \frac{1}{a_1^2}\right) \quad . \quad (2·04)$$

Hence by measuring p_{12} with a manometer, m may be estimated. A well-designed tube will usually yield a value for m that is correct to within one or two per cent.

If the effect of viscosity is to be investigated, the dimensional theory (considered below) may be used, the functions being evaluated experimentally.

The decrease in pressure on going from a_1 to a_2 can either be considered as due to an increase in kinetic energy and an equal decrease in 'pressure energy'; or it may be thought of directly as due to the fact that, for the velocity to increase towards a_2, there must be an acceleration produced by a resultant force in the direction a_1 to a_2, and the pressure must therefore be greater at a_1 than at a_2.

The Method of Dimensions ; Geometrical Similarity.—The Method of Dimensions* may be used to determine the form of the relationship between, (1) the pressure difference p_{12} between any two chosen points in a chosen system where liquid flows steadily between solid boundaries (pressure differences due to difference in level having been deducted), (2) the mass m per second crossing any transverse section (m not varying with time), (3) ρ the density, (4) μ the viscosity of the fluid, and (5) r a length that specifies the linear dimensions of the walls of the system.

Since only one length r is specified, it follows that we are assuming any change in the length to apply to all lengths

* Lord Rayleigh, *Scientific Papers*. The National Physical Laboratory, *Collected Researches*.

in the system. If the size of the tube be changed, it must be reduced or magnified throughout, being kept '*geometrically similar.*'

If no other variables are concerned we may write the relationship

$$\phi_1(p_{12}, m, r, \rho, \mu) = 0 \quad . \quad . \quad . \quad (2\cdot05)$$

The function ϕ_1 involves the variables and constants depending on the shape of the walls and the position of the two chosen points.

Any change in the units used to measure mass, length and time must leave the function unchanged in value. Hence wherever the variables occur in the function, they must occur in products that have zero dimensions when expressed in terms of mass, length and time (so that the value of the product is unchanged by any change of units).

Let $p_{12}{}^{\alpha}.m^{\beta}.r^{\gamma}.\rho^{\delta}.\mu^{\varepsilon}$ be such a product having zero dimensions. The variables have dimensions:

p_{12}	$ML^{-1}T^{-2}$
m	MT^{-1}
r	L
ρ	ML^{-3}
μ	$ML^{-1}T^{-1}$

Putting the powers to which mass length and time are raised, in the product, each equal to zero, we get

$$\left.\begin{aligned} \alpha + \beta + \delta + \varepsilon &= 0 \\ -\alpha + \gamma - 3\delta - \varepsilon &= 0 \\ -2\alpha - \beta - \varepsilon &= 0 \end{aligned}\right\}$$

Since there are three equations and five unknown quantities, two of these must be retained (say α and β). Then

$$\left.\begin{aligned} \gamma &= 2\alpha - \beta \\ \delta &= \alpha \\ \varepsilon &= -2\alpha - \beta \end{aligned}\right\}$$

Substituting these values in the indices of the product we have

$$p_{12}{}^{\alpha}.m^{\beta}.r^{(2\alpha-\beta)}.\rho^{\alpha}.\mu^{(-2\alpha-\beta)}$$

or

$$(p_{12}.r^2.\rho.\mu^{-2})^{\alpha}.(m.r^{-1}\mu^{-1})^{\beta} \quad . \quad . \quad . \quad . \quad (2{\cdot}06)$$

of zero dimensions.

By assigning values to α and β, an indefinite number of such non-dimensional products can be obtained. But they are all derivable from two independent products, such as $p_{12}.r^2.\rho.\mu^{-2}$ and $mr^{-1}.\mu^{-1}$; (or $p_{12}.r^4\rho m^{-2}$ and $mr^{-1}\mu^{-1}$, obtained respectively by putting $\alpha = 1$, $\beta = -2$ and $\alpha = 0$, $\beta = 1$); (or $p_{12}r^3\rho\mu^{-1}m^{-1}$ and $mr^{-1}\mu^{-1}$, obtained by putting $\alpha = 1$, $\beta = -1$ and $\alpha = 0$, $\beta = 1$).

(The reason for all the products being derivable from two chosen independent products is that there are five variables, but three fundamental units of mass, length and time.)

Since the function ϕ_1 can only involve these products, which are all derivable from a chosen pair, we may write equation (2·05):

$$\phi_2(p_{12}.r^4\rho m^{-2},\ mr^{-1}\mu^{-1}) = 0 \quad . \quad . \quad . \quad (2{\cdot}07)$$

or solving for the first product, taking the square root and transposing we get

$$m = r^2.\sqrt{p_{12}.\rho}\ .f_1\left(\frac{m}{r\mu}\right) \quad . \quad . \quad . \quad . \quad (2{\cdot}08)$$

By choosing instead the last pair of independent products suggested above, or by writing $f_2\left(\frac{m}{r\mu}\right) = \left\{f_1\left(\frac{m}{r\mu}\right)\right\}^2 \Big/ (m/r\mu)$, equation (2·08) may be written

$$m = \frac{r^3p_{12}.\rho}{\mu}\ .f_2\left(\frac{m}{r\mu}\right) \quad . \quad . \quad . \quad . \quad (2{\cdot}09)$$

Equations (2·08) and (2·09) show that $\frac{m}{r^2\sqrt{p_{12}.\rho}}$ and $\frac{m\mu}{r^2 p_{12}.\rho}$ do not change in value if $\frac{m}{r\mu}$ be kept constant, even if m, r and μ themselves are changed. Thus either of these products has a single value for one value of $\frac{m}{r\mu}$, and a curve may be plotted (from experiments or theory) to relate either of these and $\frac{m}{r\mu}$. This curve is general for all liquids, rates of flow, and sizes of the system; provided the system and two chosen points be kept geometrically similar.

In almost every case, when the value of $m/r\mu$ is large (a large rate of flow) the kinetic energy terms predominate, and $m^2 \propto p_{12}$ approximately. Thus f_1 in equation (2·08) tends to a constant value. Similarly, for small values of $m/r\mu$ (small rates of flow), viscous forces predominate and $\left(\frac{m}{\rho}\right) \propto \frac{p_{12}}{\mu}$. Then f_2 in equation (2·09) degenerates into a constant and no turbulence is produced.

If the size of the system cannot be specified by the single variable r, but requires more (such as the case of a circular orifice of radius r in a plate of thickness Δ, placed in a tube of radius R), the application of the above theory would give in place of equation (2·08),

$$m = r^2.\sqrt{p_{12}.\rho}\, f_1\left(\frac{m}{r\mu}, \frac{\Delta}{r}, \frac{R}{r}\right) \quad . \quad . \quad . \quad (2·10)$$

If f_1 is to remain unchanged in value we must keep (in general) $m/r\mu$ and also Δ/r and R/r unchanged in value. Hence all systems must be made *geometrically similar*, unless the effect of Δ/r and R/r on the function is about to be investigated.

Conversely, if the boundary is kept geometrically similar, and $m/r\mu$ is unchanged, the stream lines and eddies remain geometrically similar. (Unless the eddies be moving and the functions vary periodically with time.)

If desired, a pressure gradient $\frac{dp}{dx}$ at a chosen point in a chosen direction may be used instead of using p_{12}. Equation (2·08) would then take the form

$$m = r^{2\frac{1}{2}} . \sqrt{\rho . \frac{dp}{dx}} . f_2\left(\frac{m}{r\mu}\right) \quad . \quad . \quad . \quad (2·11)$$

(The pressure gradient is supposed to be that obtained after deducting that due to differences in level.)

In any of these equations, the mass passing per second may be written in terms of Q, the volume passing per second, or u, defined as the mean velocity over a transverse section or as the velocity at any chosen point in any chosen direction, may be used. The product $m/r\mu$ then is replaced by $Q\rho/r\mu$ or $ur\rho/\mu$. (This last form is more useful when the velocity of a body moving through a quantity of liquid is concerned.)

Dimensional Theory applied to an Orifice.—Applying the above theory to an orifice* situated in a wall separating two reservoirs, p_{12} may be used to represent the difference between the pressure in the two reservoirs. In place of equation (2·08) we may write

$$Q = \pi r^2 . \sqrt{2 p_{12}/\rho} . f\left(\frac{Q\rho}{r\mu}\right) \quad . \quad . \quad . \quad (2·12)$$

where Q is the volume passing per second, r is the radius of the orifice (supposed in a very thin plate), and f is now the coefficient of discharge C of equation (1·04).

* Bond, *Proc. Phys. Soc.*, Vol. xxxiii, p. 225, June 1921, and Vol. xxxiv, p. 139, June 1922.

The experimental curve relating $f\left(\frac{Q\rho}{r\mu}\right) = C$, and $\sqrt{\frac{Q\rho}{r\mu}}$ is shown in Fig. 23.

For large values of $Q\rho/r\mu$, C approximates to a value between 0·63 and 0·61, probably reaching the latter. At first, when viscosity becomes more important, the function increases. This may be due to the prevention of so much eddy production, or to a change in the coefficient of contraction. For small

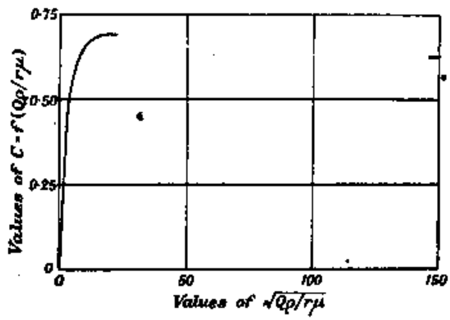


FIG. 23.

values of $Q\rho/r\mu$, the curve becomes a straight line through the origin and is represented by

$$Q = \frac{\pi}{8} \cdot \frac{p_{12}}{\mu} \cdot \frac{r^3}{(1\cdot 14 + \triangle/r)} \quad . \quad . \quad . \quad (2\cdot 13)$$

where $\triangle$ is the thickness of the plate.

This equation, which may be used in experiments to find the viscosity of very viscous liquids, is of similar form to equation (2·09), and may be compared with Poiseuille's equation (2·15).

Flow through Straight Tubes.—In the apparently simple case of liquid flowing steadily through a straight tube of

constant circular cross section, equation (2·11) shows that

$$\frac{dp}{dl} = \frac{m^2}{\rho r^5} f\left(\frac{m}{r\mu}\right) \quad . \quad . \quad . \quad . \quad (2{\cdot}14)$$

where r is the radius of the tube, and l distance along the tube. For a long tube it is evident that there is no acceleration of the liquid towards or away from the axis; and hence there is no pressure gradient in a direction across the tube, except that due to differences in level. If the part of the tube considered is near its entrance, the function f might depend on the distance from the entrance, the nature of the entrance, and the degree to which the liquid entering was disturbed previous to entrance. Also if the walls of the tube were roughened, an effect on that account might be expected.

When the rate of flow is very small (i.e. $m/r\mu$ small) the above equation will tend to the form

$$\frac{dp}{dl} \propto \frac{\mu m}{\rho r^4}$$

For such slow rates of flow the liquid is found to move in stream lines parallel to the axis of the tube, the full equation having been deduced by Poiseuille. The forces due to the pressure differences corresponding to difference in level,

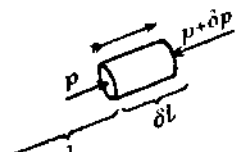


FIG. 24.

are in equilibrium with the gravitational forces, and so are supposed deducted (as in the previous calculations). Considering a cylindrical portion of the fluid (Fig. 24), with its axis coincident with that of the tube, we find a force tending

to produce motion, of amount $(-\delta p)\pi r^2$ where r is the radius of the cylinder. The viscous forces opposing motion are all parallel to the axis, acting at the outer curved surface of the cylinder. The velocity gradient along the radius, at a point on the surface of the cylinder, is $\left(-\dfrac{du}{dr}\right)$. The negative sign is inserted to obtain the positive value of the gradient, for u decreases as r increases, being zero at the walls of the tube. Thus we have

$$(-\delta p)\pi r^2 = \mu(2\pi r\delta l)\left(-\frac{du}{dr}\right)$$

or $\dfrac{\delta p}{\delta l} = \dfrac{2\mu}{r}\dfrac{du}{dr}$ constant along the tube. Hence

$$\int r.dr = \frac{2\mu}{\dfrac{\delta p}{\delta l}} . \int du$$

or

$$u = \frac{(\delta p/\delta l)}{2\mu} \cdot \frac{(r^2 - r_m^2)}{2}$$

where r_m represents the maximum value of r, the internal radius of the tube. The pressure gradient in the direction of motion, $\delta p/\delta l$ is negative.

Hence

$$\frac{m}{\rho} = \int 2\pi r u dr = \frac{\pi}{2\mu} \cdot \frac{\delta p}{\delta l} \cdot \int_{r=0}^{r=r_m} (r^3 - r.r_m^2)dr = -\frac{\pi}{8}\frac{r_m^4}{\mu}\frac{\delta p}{\delta l}$$

Or, omitting the subscript, we have

$$\frac{dp}{dl} = -\frac{8\mu}{\pi r^4}\frac{m}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2{\cdot}15)$$

The velocity variation over the diameter is seen to be parabolic; and the maximum velocity at the centre is equal to twice the average velocity obtained by dividing the volume

passing per second by the area of the transverse section. When the rate of flow is large, and $m/r\mu$ is large, the flow in the tube is found to become turbulent, except near the walls, and the pressure gradient is larger than that given by equation (2·15). Until eddies begin to be produced, Poiseuille's equation (2·15) is found to hold exactly, except near the entrance to the tube. The production of turbulence was investigated by Osborne Reynolds* with an apparatus illustrated in Fig. 25. The nature of the flow was investigated by means of a coloured filament of liquid emitted from a small tube near the entrance to the tube in which the main flow took place. It was found that for large rates of flow

FIG. 25.

turbulence was set up some distance along the tube. As the rate of flow increased, the point at which eddies were formed moved towards the entrance. Apart from the effect of nearness to the entrance cone, it is found that turbulence generally commences at approximately the same value of $m/r\mu$. A transition then occurs, and at higher rates dp/dl is approximately proportional to the square of the rate of flow. In smooth tubes the power varies from about 1·75 towards 2·00 as the rate of flow is increased. In tubes the inner surface of which is artificially roughened (by cutting grooves, for instance) the power reaches the value two.† If the walls

* *Phil. Trans. Roy. Soc.*, 1883.
† Stanton, *Proc. Roy. Soc.*, 85A, p. 366, 1911.

of the tube at some point near the entrance be heated, the rate at which heat is communicated to the fluid is found to increase when turbulence begins. The eddies produce a large convection of heat from the walls. By inserting a sensitive thermometer into the tube further along, its sudden rise in reading may be used to detect the commencement of turbulence,* in place of observing coloured filaments, or measuring the pressure gradient.

Two points of transition (*critical velocities*) may be considered. For any eddies present in the first part of the tube (due to intentional disturbance of the liquid before entrance) just to be damped down, and the flow to take place in steady stream lines further along the tube, the value of $m/r\mu$ is fairly definite. On the other hand, the value of $m/r\mu$ for eddies just to be produced, when the flow at the commencement is in stream lines, is not very definite. Any small disturbance present, that does not appear to produce disturbance of the stream lines near the entrance, has an effect on the value of $m/r\mu$ for which turbulence is produced further along.

The 'lower critical velocity' when eddies present are just damped down, is found for smooth tubes to be given approximately by

$$\frac{m}{r\mu} = 3100 \quad \text{or} \quad u_{(mean)} = \frac{\mu}{r\rho} \cdot 1000 \quad . \quad . \quad (2{\cdot}16)$$

One noticeable difference between this equation and any of those previously considered, is that the constant has the large value of about a thousand.

When the critical speed is exceeded, the velocity is found still to be small near the walls. In order to investigate the flow at very small distances from the walls with a Pitot tube, the latter must be exceedingly small. Dr. Stanton† employed a

* Barnes and Coker, *Proc. Roy. Soc.*, Vol. 74A, p. 341 (1905).
† *Proc. Roy. Soc.*, Vol. 97A, p. 413 (1920).

Pitot tube placed so that one side of its opening was flush with the side of the tube. The velocity estimated by the Pitot tube was that at some unknown point between the wall and the other edge of the opening. It has been shown since that the effect of viscosity has also to be taken into account.* In order to interpret the readings obtained with the Pitot tube, it was calibrated in position in a tube in which stream line flow was taking place, and the velocity was therefore known. The result of these experiments showed that the

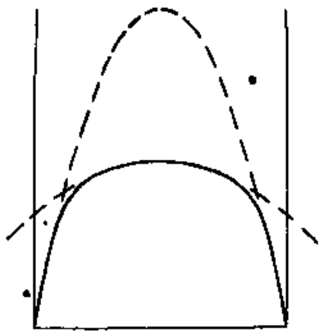


FIG. 26.

velocity increased almost in direct proportion to the distance from the walls for a small distance, the flow being presumably in stream lines in this region. The velocity distribution over this part is represented by a portion of a parabolic curve, such as occurs when the flow is in stream lines even at the axis. In the central portion, where eddies are being produced, the curve is approximately part of a larger parabola, the two curves being joined by a transition layer (Fig. 26). The whole curve found varies in shape with the value of $m/r\mu$.

* Barker, *Proc. Roy. Soc.*, Vol. 101A, p. 435 (1922).

Figure 27* represents values of $-f\left(\frac{m}{r\mu}\right)$ of equation (2·14) plotted against $m/r\mu$. For small values of $m/r\mu$

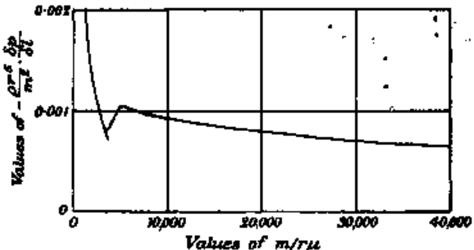


FIG. 27.

the product of these two is equal to $\frac{8}{\pi}$, and the curve is a rectangular hyperbola (2·15). At higher rates the state of transition is reached: and for yet large values of $m/r\mu$

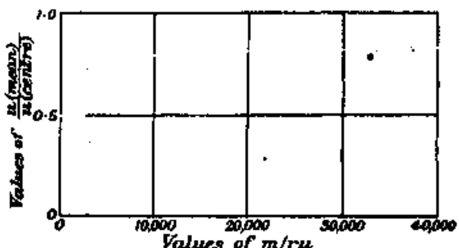


FIG. 28.

the ordinates change only slowly, but continue to decrease at the highest rates so far attained. In Fig. 28 the ratio of the mean velocity over the transverse section, to the

* Stanton and Pannell, *Phil. Trans. Roy. Soc.*, 214A, p. 199 (1914).

maximum velocity at the centre, is plotted against values of $m/r\mu$. When the flow is in stream lines the ordinate is equal to 0·5.* After the transition it reaches a higher value, and continues to increase slowly. All these diagrams refer to points far from the entrance, and to smooth-walled tubes. Subject to these conditions, any experiment should give results agreeing with the curves. (In place of dp/dl, the tangential force on unit area of the walls (R), may be used. The relationship between these alternative variables is $R = -\frac{r}{2}\frac{dp}{dl}$).

The question that naturally arises is, what is the mechanism of the phenomenon. It appears that eddies exist in the more central parts of the tube, being formed at a small distance from the walls; and that the radial migration of liquid produced by the eddies tends to equalise the velocities parallel to the axis, all over the central turbulent portion. The velocity still varies with the distance from the axis, there being a tangential force acting on the outer surface of any cylinder such as illustrated in Fig. 24. This force is produced not by pure viscous stream line flow, but by the *transfer of momentum due to the radial velocity of the eddies*. The liquid has a kind of pseudo-viscosity* due to the eddy motion. The transfer of momentum is much greater where turbulence occurs, and hence the radial velocity gradient corresponding to a given force is smaller than when stream line flow occurs. Thus the central part of the curve indicated in Fig. 26 is less pointed than the curve that occurs when the motion is stream line. Many attempts have been made to calculate theoretically at what value of $m/r\mu$ turbulence should commence, or just die down.† The peculiar nature of the problem is understood when the large value of the constant is remembered. An algebraic constant of the order of a thousand would

* G. I. Taylor, *Phil. Trans.*, 215A, 1915.
† Heisenberg, *Ann. d. Physik*, 74.7, p. 577, July 1924.

have to occur. "It is perhaps not without significance to notice that constants approaching this in magnitude occur in problems of the bending and stability of beams. In the present case the stability of a tube of flow is concerned."

Other Applications of Dimensional Theory.—The dimensional theory can be applied to a large number of cases of fluid flow through tubes, such as the effect of bends in the tube,* the effect of the tube being tapering,† the effect of a sudden change in diameter of the tube at some point, the effect of roughening the inner surface of the tube,‡ and to questions of heat convection § and periodic production of eddies.‖ In most cases the values of the functions can only be obtained experimentally at present.

* Escande and Ricaud, *Rev. Gén. d'Él.* 15, p. 723, April 1924.
† Bond, *Proc. Phys. Soc.*, Vol. 34, p. 187, Aug. 1922.
‡ Stanton, *Proc. Roy. Soc.*, 85A, p. 366, 1911.
§ *The Mechanical Properties of Fluids*, p. 179 (Blackie).
‖ *Dict. of Applied Physics*, Vol. 1, Dynamical Similarity, p. 89.

CHAPTER III

SURFACE WAVES: AND MOTION IN CHANNELS AND OVER WEIRS

Types of Waves.—Surface waves on liquid are due to the downward force of gravity acting on any part of the liquid momentarily raised above the general level of the surface, and also to the downward force due to the surface tension over the convex surface formed (and similar upward forces for a momentary depression). The waves may be on liquid of a depth comparable with the wave-length and wave-depth; or the depth of liquid may be very great. The waves may be rectilinear, or radiating from a centre; moving or stationary; of constant or varying contour; and frictional forces may produce damping. Only a few simple cases will be considered here. In most waves, the motion of any particle is oscillatory, no motion of the liquid as a whole relative to the boundaries taking place. Viscosity causes the liquid near the walls to be almost at rest, and produces a periodically varying tangential force at the walls. Eddies may be produced near the boundary; and damping of the wave motion occurs, the chief effect of which is to decrease the amplitude. But the solid boundaries have generally a much smaller effect than in cases such as the flow through tubes and orifices, and the motion can be considered approximately irrotational and non-viscous. Many cases of wave motion are thus capable of detailed calculation.

* Lamb, *Hydrodynamics.*

Waves on Shallow Liquid.—Firstly let us consider a single long plane wave on shallow liquid (Fig. 29). If the observer be supposed to move along with the wave, the problem is the same as the steady stream-line flow of the liquid between the base and the curved surface of the wave. If the velocity be assumed the same at all points on a vertical line, we have $u(d + z)\rho = c.d.\rho$ where u is the velocity of the liquid relative to the observer, where the depth is $(d + z)$, and c is the observed velocity of the undisturbed liquid, of depth d (both being in the direction XO).

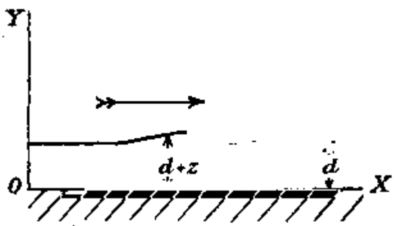


FIG. 29.

Applying Bernoulli's equation to the surface layer, and denoting the height above the level of the undisturbed surface by z, we have

$$\tfrac{1}{2}\rho c^2\left(\frac{d}{d+z}\right)^2 + \rho g z = \text{constant} = \tfrac{1}{2}\rho c^2,$$

or
$$c^2 = \frac{2gz}{1-\left(\dfrac{d}{d+z}\right)^2} = gd\left\{1 + \frac{3}{2}\frac{z}{d} - \ .\ \ .\ \right\}$$

Hence, when z/d is small compared with unity, the velocity of the wave is

$$c = \sqrt{gd}\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (3{\cdot}01)$$

and is independent of the wave length. But if the elevation

above the undisturbed surface be comparable with the depth of liquid, the velocity is greater for greater elevations. Thus the highest part of the wave travels fastest, and the shape changes, the wave becoming high in front. When the amplitude is small and the wave form almost unchanging, the assumption that the velocity is the same at all points on a vertical line is evidently sufficiently accurate. The horizontal velocity of the fluid, relative to the ground, in the direction of motion of the wave, is $c - u = c\left(1 - \frac{d}{d+z}\right) = \frac{cz}{d}$ approximately, at all points on a vertical line where the wave elevation is z. The upward displacement of a particle originally at a height y_o above the bottom surface, is zy_o/d approximately; and the upward velocity of the particle is $(y_o/d) \cdot \frac{dz}{dt}$. Considering simple harmonic waves of small amplitude, we may write $z = a \cos 2\pi\left(\frac{t}{\tau} - \frac{x_o}{\lambda}\right)$. This represents a wave motion of period τ, and wave length λ. And since an increase in t of amount τ has the same effect on z as a decrease in x_o of amount λ, the wave is moving in the positive direction of the axis OX.

The upward velocity, and vertical amplitude, at a point, are therefore respectively

$$-\frac{y_o}{d}\frac{2\pi}{\tau} a \sin 2\pi\left(\frac{t}{\tau} - \frac{x_o}{\lambda}\right) \text{ and } \frac{y_o}{d} \cdot a$$

The velocity in the direction of wave motion OX, and the horizontal amplitude, at the same point are

$$\frac{c}{d} \cdot a \cdot \cos 2\pi\left(\frac{t}{\tau} - \frac{x_o}{\lambda}\right) \text{ and } \frac{c}{d} \cdot \frac{\tau}{2\pi} \cdot a.$$

Every particle therefore partakes of two simple harmonic motions at right angles, of equal periods, the horizontal

oscillation lagging behind the vertical in phase, by an amount $\pi/2$. The particles thus describe ellipses (in a clock-wise or negative direction in the diagram), the major axes of which are horizontal. The ellipse degenerates into a horizontal straight line for points on the base. For a particle on the surface, the ratio of the vertical and horizontal amplitudes or maximum velocities is $2\pi d/\lambda$. Since the vertical component of velocity was not considered in applying Bernoulli's equation, it is clear that equation (3·01) (and the calculated velocities of the particle) might be incorrect when the energy of vertical motion became comparable with that of horizontal motion. Hence d^2/λ^2 must be supposed small compared to unity.

The above form of wave can be experimented upon by obtaining stationary waves in a long horizontal trough. The wave length of the fundamental oscillation is equal to twice the length of the trough. The depth of liquid, and period of the waves may be measured, and the validity of equation (3·01) tested.

Waves on Deep Liquid.—When waves on the surface of deep liquid are concerned, the horizontal velocities are not influenced by the difference in the depth of liquid below the crest and trough of a wave. It is now necessary to consider vertical velocities as well as horizontal ones. Let us assume as the simplest case that any particle at the surface partakes of horizontal and vertical simple periodic motions simultaneously. For the simplest case the periods of the two motions would clearly be equal to one another and to that of the wave. The nature of the displacements is indicated by arrows under Fig. 30. It is evident that the horizontal displacement lags in phase $\pi/2$ behind the vertical. The position of a particle at a time t may therefore be written

$$x = x_0 + a \sin 2\pi\left(\frac{t}{\tau} - \frac{x_0}{\lambda}\right) \text{ and } y = b \cos 2\pi\left(\frac{t}{\tau} - \frac{x_0}{\lambda}\right)$$

The wave is represented as moving in the direction OX, and the period is τ and the wave length λ. The axis of OX has been chosen so as to be mid-way in level between the crests and troughs. It will be seen later that this level is not that of the undisturbed liquid.

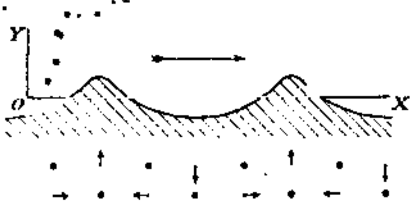


FIG. 30.

Let the observer travel with the same velocity as the wave, $c = \frac{\lambda}{\tau}$; and then apply Bernoulli's equation to the surface layer. The observed horizontal velocity at a point (x, y), is

$$\frac{dx}{dt} - c = \frac{2\pi}{\tau} a \cos 2\pi\left(\frac{t}{\tau} - \frac{x_0}{\lambda}\right) - \frac{\lambda}{\tau}$$

and the vertical velocity

$$\frac{dy}{dt} = -\frac{2\pi}{\tau} b \sin 2\pi\left(\frac{t}{\tau} - \frac{x_0}{\lambda}\right)$$

Hence

$$\rho g b \cos 2\pi\left(\frac{t}{\tau} - \frac{x_0}{\lambda}\right) + \tfrac{1}{2}\rho\left[\left\{\frac{2\pi}{\tau} a \cos 2\pi\left(\frac{t}{\tau} - \frac{x_0}{\lambda}\right) - \frac{\lambda}{\tau}\right\}^2 + \left\{-\frac{2\pi}{\tau} b \sin 2\pi\left(\frac{t}{\tau} - \frac{x_0}{\lambda}\right)\right\}^2\right] = \text{constant}$$

(This constant cannot be written equal to that obtained for the undisturbed liquid, because a definite amplitude

is being considered, and the transition to level liquid would correspond to a gradual change to zero amplitude. Also the difference in level between OX and that of the undisturbed liquid surface is not yet known.)

For this expression to have the same value for all values of $\left(\frac{t}{\tau} - \frac{x_0}{\lambda}\right)$, we must have $a = b$, and $-\rho g b + \frac{1}{2}\rho\left(\frac{4\pi a \lambda}{\tau^2}\right) = 0$ or

$$c = \frac{\lambda}{\tau} = \sqrt{\frac{g\lambda}{2\pi}} \quad . \quad . \quad . \quad . \quad (3{\cdot}02)$$

Since Bernoulli's equation is then exactly satisfied at all points on the surface (viscosity and surface tension being neglected) it is evident that the amplitude need not be considered small.

The surface particles move in circles, with constant angular velocity. The angular position at any time varies directly with the distance of the particle's mean position from the origin. Thus the contour of the wave at any instant is such as is described by a point on a wheel rolling along a horizontal line, above the liquid, but parallel to its surface (a trochoid). Nothing has been said about the position of the particles below the surface. An inspection of the trochoid section of the wave shows that the mean position about which a surface particle oscillates is above the surface of the undisturbed liquid. Particles below can be shown to move in circles of smaller radius, with the same angular velocity.

The Effect of Surface Tension ; Ripples.—A wave of sine form and of small amplitude may be represented at any chosen instant by $y = a \sin (2\pi x/\lambda)$, (where λ is the wave length).

The vertical restoring force acting on a length of wave δx of unit width is

$$-\mathrm{T}(\sin\alpha_2 - \sin\alpha_1) = -\mathrm{T}\left[\left\{\frac{dy}{dx} + \frac{d}{dx}\left(\frac{dy}{dx}\right)\delta x\right\} - \frac{dy}{dx}\right]$$

$$= -\mathrm{T}\frac{d^2y}{dx^2}\delta x$$

where tangents of the angles (α_1, α_2) the wave surface makes with the horizontal have been used as a close approximation to the sines of the angles (see Fig. 31). Thus the restoring force due to surface tension acting on the section is equal to $\mathrm{T}\,.\,\frac{4\pi^2}{\lambda^2}\,.\,\delta x\,.\,y$. The restoring force due to the action of gravity is equal to $\rho g.\delta x.y$ for the section.

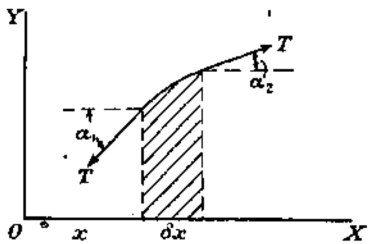


FIG. 31.

Hence the action of surface tension (which only changes the restoring force and potential energy and not the kinetic energy) is equivalent for such sine waves of small amplitude to an increase in g of amount

$$\frac{4\pi^2\mathrm{T}}{\lambda^2\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}03)$$

It is evident that surface tension becomes important for small values of λ (the curvature of the surface becoming relatively large). The waves are then often described as ripples. In the case of shallow water waves, the velocity will thus increase

as the wave length becomes smaller. If the depth of water be still small compared to the wave length, the amplitude small compared to the depth, and the wave form approximately a sine curve, we have

$$c = \sqrt{\left(g + \frac{4\pi^2 T}{\lambda^2 \rho}\right) d} \qquad (3.04).$$

In the case of deep water waves, when the amplitude is small, and the wave is almost of sine form, we have similarly

$$c = \sqrt{\frac{\lambda}{2\pi}\left(g + \frac{4\pi^2 T}{\lambda^2 \rho}\right)} \quad . \quad . \quad . \quad . \quad (3.05)$$

The velocity is then large both for very large and very small wave lengths. For the velocity to be a minimum, λ must be such that $\lambda g + \frac{4\pi^2 T}{\lambda \rho}$ is a minimum. This is clearly the case when $g\lambda = \frac{4\pi^2 T}{\rho} \cdot \frac{1}{\lambda}$ or $\lambda = \sqrt{\frac{4\pi^2 T}{g\rho}}$. For water at ordinary temperature, the minimum velocity is about 23 cm. per second, and the wave length about 1·7 cm.

Groups of Waves; Group Velocity.—If two trains of waves of nearly equal periods and wave lengths, and of comparable amplitude be superposed, the phenomenon of "beats" is observed. If the amplitudes are equal a series of groups of waves is formed, separated by almost undisturbed liquid. The wave trains are in phase at the centre of each group. The frequency of the groups is equal to the difference between the frequencies of the separate waves.

Thus we may write

$$c = n\lambda \qquad c' = n'\lambda' \qquad N = n - n' = \frac{c}{\lambda} - \frac{c'}{\lambda'}$$

where N, n, n' are the respective frequencies. The numbers of

small waves equal to the distance L between consecutive groups are L/λ and L/λ'. These must differ by unity to obtain waves in phase at each group. And (unless c changes more rapidly than in direct proportion to λ, and the groups travel in the opposite direction to the separate waves) we have $\frac{L}{\lambda} - \frac{L}{\lambda'} = 1$. Hence the velocity of the group, U, is given by

$$U = \frac{N}{1/L} = \frac{c/\lambda - c'/\lambda'}{1/\lambda - 1/\lambda'} = \frac{c/\lambda - \dfrac{c + \delta c}{\lambda + \delta\lambda}}{\dfrac{1}{\lambda} - \dfrac{1}{\lambda + \delta\lambda}}$$

or

$$\frac{U}{c} = 1 - \frac{\lambda}{c}\frac{dc}{d\lambda} \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}06)$$

The rate at which energy passes any plane normal to the direction of the wave, clearly depends on the group velocity, and not directly on that of the individual waves.

In the case of waves on deep liquid, where $c \propto \lambda^{1/2}$, this shows that $\frac{U}{c} = \frac{1}{2}$, or the group velocity is half that of the waves. Waves appear at the back of the group, move along the group their amplitude increasing, and finally die out again at the front of the group.

Surface Waves produced by a Solid Body.—When a large body on the surface of liquid, such as a ship, moves relatively to the liquid, various sets of waves are produced.* Three kinds may be mentioned here. The oblique bow wave produces a V-shaped group of waves, the apex being at the bow and the limbs trailing behind. Each limb consists of a series of waves in step-like formation, the fronts of individual waves being more nearly normal to the direction of motion of the ship than are the limbs of the V. These may be called

* See Lord Kelvin, Ship Waves, *Popular Lectures*, Vol. iii.

echelon waves. A similar set of waves is formed by the stern of the ship. A third set of waves is formed at the sides of the ship, owing to the curvature of the stream lines, and the corresponding excess pressure at the ends and reduced pressure towards the centre of each side.* These waves are known as transverse waves, and move with the same velocity as the ship. These waves may produce a series of waves in their wake.

If the body be small (such as a stake placed in a stream), the effect of surface tension predominates. Small waves, or ripples, travel away from the body, their velocity being due to their motion relative to the liquid, together with the motion of the liquid itself. Any ripples that are formed in front of the body would appear at rest, if their velocity be equal and opposite to the velocity of the stream. The stream velocity will be zero at the surface of the body, and increase with distance from the body. At a point where the velocity equals the minimum velocity of the waves (if the velocity of the stream is as large as this), stationary waves of this critical wave length will be formed. Further up the stream, where the velocity is larger, shorter stationary waves will be formed. If their wave length at an appreciable distance from the body can be found, their velocity may be deduced from the known value of surface tension, the acceleration due to gravity, and the density of the liquid; and this velocity is approximately the velocity of the surface of the stream at a distance from the body.

Similarly stationary waves or ripples may be observed, when a jet of liquid impinges on a flat plate and spreads out into a flat sheet of liquid, moving radially outwards on the surface of the plate.

Ripples may be formed by attaching a plate to a tuning-fork, so as to dip into the surface of liquid in a trough. The

* See p. 67.

ripples may be made to appear stationary by illuminating them by an intermittent beam of light that shines once every vibration of the fork, or by observing them through an aperture that is opened by means of the fork once each oscillation. The wave length may be measured and used to determine the surface tension of the liquid.* Special precautions have to be taken to avoid reflection of the ripples at the edge of the dish; and to prevent the vibration of the dish causing other ripples.

The oscillatings in a jet of liquid may be used to compare surface tensions; † and the method of dimensions may be used to investigate the conditions of stability of the jet.‡

Flow along a Channel the Base of which contains an Elevation.—Let x (Fig. 32) be the elevation of any part

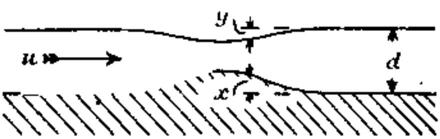


FIG. 32.

of the obstruction, y the depression of the liquid surface, u the velocity of the liquid and d its depth before it reaches the obstruction. The calculation (neglecting viscosity and surface tension) is similar to that of a wave in shallow liquid. Considering unit width of channel we have for the surface layer

$$\tfrac{1}{2}\rho u^2 + dg\rho = \tfrac{1}{2}\rho u^2 \left\{ \frac{d}{d - x - y} \right\}^2 + (d - y)g\rho,$$

* Lord Rayleigh, *Scientific Papers*. Dorsey, *Phil. Mag.*, xliv, p. 369 (1897).

† Pederson, *Phil. Trans.* 207A, p. 341 (1908).

‡ S. W. J. Smith and H. Moss, *Proc. Roy. Soc.*, Vol. 93A, p. 373 (1917).

or

$$2yg = u^2\left\{\left(\frac{d}{d-x-y}\right)^2 - 1\right\} = \frac{u^2(2d-x-y)(x+y)}{(d-x-y)^2}$$

or approximately $u^2 = \dfrac{ygd}{x+y}$ when x and y are small compared to d.

The arrangement is like a Venturi tube open to the atmosphere; and it may be used to measure u.

The 'V' Notch.—Many forms of 'notch' or 'weir' are used for measuring the mass of water flowing along channels. Two kinds only will be considered here. The 'V' notch consists of an erect V-shaped notch or slot cut in the upper edge of a vertical thin metal sheet, that forms one side of a reservoir containing liquid (*Fig.* 33). The difference in level between

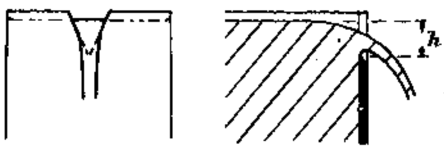


FIG. 33.

the liquid surface in the reservoir at a distance from the notch, and the tip of the V, depends on the volume of liquid flowing through per second. The relationship between this height h, and the quantity passing per second depends on the value of g. Since both the kinetic and potential energies are proportioned to ρ, the volume passing per second does not depend on the density, unless viscous or surface tension forces are considered. Denoting the angle of the notch by θ, we may consider first the variables

m	MT^{-1}
h	L
θ	—
ρ	ML^{-3}
g	LT^{-2}

We then find that

$$m = \rho g^{\frac{1}{2}} h^{2\frac{1}{2}} . f(\theta) . \quad . \quad . \quad . \quad . \quad (3{\cdot}07)$$

If viscosity and surface tension have to be considered, the function becomes

$$f\left(\theta, \frac{m}{h\mu}, \frac{T}{h^2 \rho g}\right)$$

For cases generally occurring, equation (3·07) is found to be sufficiently accurate. Thus, a single constant is required for any chosen angle of notch. It is found that $f(\theta) = 0{\cdot}447 \tan\frac{\theta}{2}$ approximately. The jet is supposed to emerge without wetting and running down the outer face of the wall. If any obstruction is placed in the reservoir, or if the V is near to the walls or base, the function will not have its normal value.

Rectangular Notch or Weir.—Another device for measuring the volume passing per second is a rectangular notch or

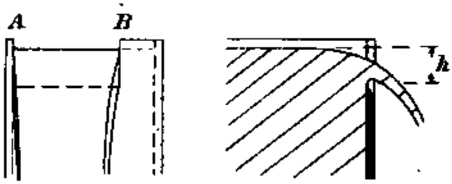


FIG. 34.

weir. In this case the liquid flows over the horizontal upper edge of a thin wall of the reservoir, the sides of the weir being formed by the side walls of the reservoir as at A (Fig. 34).

Or a rectangular notch may be cut in the upper edge of the wall, so that the sides of the notch are as at B. Neglecting the effect of the side walls or edges, we see that the mass passing per second is proportional to the width w. Hence the variables concerned are m/w, h, ρ, g, μ and T. We then obtain

$$\frac{m}{w} = \rho g^{\frac{1}{2}} \cdot h^{1\frac{1}{2}} \cdot f_1\left(\frac{m/w}{\mu}, \frac{\mathrm{T}}{h^2 \rho g}\right) \quad . \quad . \quad . \quad (3{\cdot}08)$$

In many cases that occur the effects of variation of μ and T may be neglected. But it is necessary to consider the effect of the sides. We have therefore to consider the equation

$$m = w\rho g^{\frac{1}{2}} h^{1\frac{1}{2}} f_2\left(\frac{h}{w}\right) \quad . \quad . \quad . \quad . \quad (3{\cdot}09)$$

If the weir or notch is wide, so that the peculiar flow at the sides does not sensibly affect that at the central parts, we should expect the flow to be equal to the sum or difference of two terms, one of which was directly proportional to the width, and the second independent of the width. Hence we may write

$$\left.\begin{aligned} m &= \mathrm{C}_1 w\rho g^{\frac{1}{2}} h^{1\frac{1}{2}} + \mathrm{C}_2 \rho g^{\frac{1}{2}} h^{2\frac{1}{2}} \\ &= w\rho g^{\frac{1}{2}} h^{1\frac{1}{2}}\left(\mathrm{C}_1 + \frac{\mathrm{C}_2}{\mathrm{C}_1}\frac{h}{w}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (3{\cdot}10)$$

C_2 is negative, since the side effects cause a decrease in the rate of flow. Also C_2 is considerably larger in the case B above, since the stream lines then curve inwards at the sides. Variations of μ and T are again not found to affect the values of C_1 and C_2 appreciably for the cases generally occurring. C_1 is generally about 0·58 ; $\mathrm{C}_2/\mathrm{C}_1$ for sides as at B, is about $-0{\cdot}2$; and for sides as at A it may be neglected.)

CHAPTER IV

THE MOTION OF A BODY THROUGH FLUID

Force to Maintain Steady Motion.—The simplest case of the steady motion of a solid body through fluid in which it is immersed, is when the fluid is at rest or in steady stream line motion. (If eddying occurs at points distant from the moving body, it might affect the problem, and it would be necessary to know details about this eddying.)

By the method of dimensions, it is evident that the force F required to maintain the uniform velocity u relative to the fluid is given by .

$$\frac{F}{u^2r^2\rho} = f\left(\frac{ur\rho}{\mu}\right) \quad . \quad . \quad . \quad . \quad . \quad (4{\cdot}01)$$

when compressibility has no appreciable effect (i.e. motion through liquid, or slowly through a gas). The linear dimensions of the body are represented by r; and it is supposed that the body does not rotate. The function would be different for different directions of motion, unless the body were spherical. For small values of $ur\rho/\mu$ the equation tends to the form

$$F = Aur\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad (4{\cdot}02)$$

where A is a constant depending on the shape of the body. The flow is then in steady stream lines round the body. Viscous forces act, and the exact shape of the stream lines may change slightly with $ur\rho/\mu$, except when the latter is

very small. Thus A will change but little, and will asymptote to a constant value as $ur\rho/\mu$ becomes small.

The Motion of a Sphere.—Stokes* has shown that for a sphere, $A = 6\pi$, when r denotes the radius of the sphere. This result may be used to determine the viscosity, by finding the final velocity attained by a small solid sphere falling through undisturbed fluid. The force acting is given by $F = \frac{4}{3}\pi r^3(\rho' - \rho)$ where ρ' is the density of the solid. If the sphere be arranged to fall through liquid contained in a vertical tube, the effect of the walls and ends has to be taken into account.†

If the sphere be held by a fine support in a tube of square section, of side D, the velocity of the fluid at which eddies are first produced behind the sphere is given by‡

$$u = \frac{\mu}{\rho r}\left\{4{\cdot}07 + 34{\cdot}1\left(\frac{2r}{D}\right)^{3/2}\right\} . \quad . \quad . \quad (4{\cdot}03)$$

Hence, at higher velocities than those given by $u = 4{\cdot}07\,\mu/\rho r$, the equation due to Stokes, of the form of equation (4·02), can certainly no longer be used.

If the fluid be assumed to have zero viscosity, the velocity and pressure at any point near the sphere can be calculated. Then no energy is supposed to be dissipated in the form of heat, and no force would be required to keep the sphere in uniform motion. (Though in changing the velocity, energy would be required to change the kinetic energy of the liquid near the sphere.) The stream lines and pressure at the surface of the sphere, when $ur\rho/\mu$ is large, are rather similar to those calculated by the above theory, for the part of the sphere meeting the fluid. But the experimental and theoretical results differ

* *Scientific Papers* (Vol. 3).

† Ladenburg, *Ann. der Physik*, 1907.

‡ Nisi and Porter, *Phil. Mag.*, Vol. xlvi, p. 755, Nov. 1923.

considerably for points behind the sphere. In Fig. 35 the lines of flow are indicated for one value of $ur\rho/\mu$. It is seen that an eddy in the form of a ring is formed behind the sphere. (The eddy appears in section in the diagram.)

In Fig. 36 the calculated pressure at the surface of the sphere is indicated by a dotted line, and the observed pressures for a particular value of $ur\rho/\mu$ are indicated by the solid line. (The values are plotted radially, the larger circle representing the pressure in the liquid when the sphere is not inserted.) A larger pressure is found at the front of the sphere, due to outward and forward acceleration of the liquid. At the sides a reduced pressure is found, corresponding to the inward

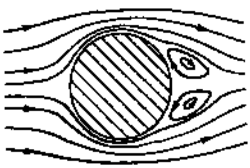


FIG. 35.

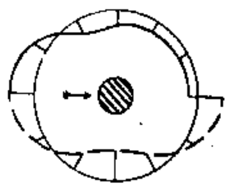


FIG. 36.

acceleration of the liquid. But this acceleration and reduction of pressure are less than calculated by the theory that neglects viscosity. The eddy consists of liquid in relatively slow motion; and the stream lines outside it do not begin to become concave as viewed from outside until they are some distance away from the sphere. Thus there is not an outward acceleration or excess pressure till a point is reached towards the rear of the eddy. The pressure at points on the surface of the sphere towards the back is thus less than the normal; and not in excess, as would be the case if the eddy were not present. Besides the forces required to accelerate the fluid, forces due to viscosity have to be considered. These forces would cause the pressure to be less on the back of the sphere.

Considering the forces acting on the sphere, it is seen that the forces in front (opposing motion) are greater than when it is at rest; and the forces on the back (in the direction of motion) are less than when at rest. The resultant is equal to F opposing the motion. The energy is dissipated in the form of heat in the stream lines and eddy. For large values of $ur\rho/\mu$ the eddy is not stationary in position; but a series of eddies is formed, each drifting off in the wake of the sphere in turn. In this case a larger force (and dissipation of energy) may be expected, tending to the form

$$F \propto u^2r^2\rho$$

Many experiments on this subject have been carried out,* but the initial turbulence of the surrounding fluid and possible rotation of the sphere make it difficult to arrive at accurate results.

The Motion of a Cylinder.—In the case of the motion of a long cylinder relative to the fluid, in a direction at right angles to the axis of the cylinder, it has been found † that eddies may occur when $ur\rho/\mu > 1{\cdot}33$. When $u\rho/\mu$ is equal to about 75, periodic eddy motion has been observed.‡ This periodic motion or production of eddies is supposed to be the cause of the 'Æolian Tones' that may be heard when a wire is placed in a transverse air current. By experiments on the notes thus produced, periodic eddy motion has been estimated § to occur when $ur\rho/\mu$ is greater than 16·5. The frequency of the motion is found to be almost proportional to u/r, and almost independent of μ/ρ. In Fig. 37, eddies produced behind a cylinder are illustrated.‖ And in Fig. 38 ¶

* L. F. Richardson, *Proc. Phys. Soc.*, Vol. 36, p. 67, Feb. 1924.

† Nisi and Porter, *Phil. Mag.*, Vol. xlvi, p. 757, Nov. 1923.

‡ *Reports and Memoranda, Advis. Com. for Aeronautics*, No. 332 (1917).

§ E. G. Richardson, *Proc. Phys. Soc.*, Vol. 36, p. 153, Apr. 1924.

‖ *Dict. of Applied Physics*, Vol. v, Model Experiments, p. 191.

¶ *Reports and Memoranda, Advis. Com. for Aeronautics*, No. 40, March 1911; No. 74, March 1913; No. 102.

the logarithm of the function f of equation 4·01 (where F is the force exerted on a length of the cylinder equal to one radius) is plotted against values of $\log_{10}(ur\rho/\mu)$. It will be noticed that a peculiarity in the curve occurs at considerably larger values of $ur\rho/\mu$ than those at which eddying commences.

FIG. 37.

As in cases such as flow through an orifice and the motion of a sphere, it is found that as $ur\rho/\mu$ is decreased from a very large value, the first effect is to cause a smaller dissipation of energy than if f were constant. This may be due to the action of viscosity damping down the eddy motion. As

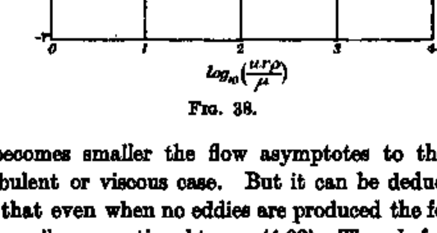


FIG. 38.

$ur\rho/\mu$ becomes smaller the flow asymptotes to the purely non-turbulent or viscous case. But it can be deduced from Fig. 38 that even when no eddies are produced the force F is not necessarily proportional to $ur\mu$ (4·02). Thus, before eddies are able to be formed, the kinetic energy of the fluid causes

the contour of the stream lines to differ slightly from those that occur at infinitely slow velocities.

The Production of Eddies.—It has already been mentioned that the moment of momentum of an eddy can only be changed by the action of tangential viscous forces at its boundary. It can also be shown that the eddy filament can only terminate on a boundary surface; or the eddy form a ring, so that the filament has no ends. For instance, if a straight eddy filament of circular section be stretched longitudinally, and no tangential forces act at its surface, its moment of momentum, $I\omega$, remains constant (where I is its moment of inertia and ω its angular velocity about its axis). The volume will remain constant, r and I will decrease, and ω increase; and hence the kinetic energy $\frac{1}{2}I\omega^2$ will increase. Since energy must have been supplied to the eddy, it will be concluded that the pressure at any possible end to the eddy could not be the same as that of the surrounding non-rotating fluid; and the eddy thus cannot terminate in this fluid.

In general, when stationary eddies are being maintained, or when eddies are being formed periodically and are decaying gradually in turn, a considerable dissipation of energy into heat occurs. The resistance experienced to the motion of a body through fluid will thus be small if the shape of the body be such as to produce little eddy motion. The conditions under which eddies may occur have been determined experimentally; but little detailed theory has been evolved that is capable of explaining the results. It is known that for small values of $ur\rho/\mu$, eddies are not produced. And the critical value of this non-dimensional product is very approximately of the order unity in the cases of the sphere and cylinder mentioned above. When streams of different forms are considered, it is found that, as a rule, eddy motion is rapidly damped down where the fluid is being accelerated—as in the case of a converging conical tube, or in front of a

moving body. But where the fluid is subsequently decreasing in velocity, eddies are often formed—as in a diverging tube or behind the moving body. When such eddying is not desirable, it may be minimised by making the decrease in velocity as gradual as possible. For this reason a long exit cone is used in a Venturi tube* (Fig. 22). Similarly, the shapes of torpedoes, airships and of the sections of various struts used in aeroplane work, is made rather blunt in front and gradually tapering in the wake.* For instance, if a bar of circular section be replaced by one having a section such as has been described, the minor axis of which is the same as the diameter of the circular bar, but the major axis about six times as large, it is found that the resistance experienced when the fluid impinges on the blunt edge, is only a seventh of that experienced by the circular strut.†

Applications in Aeronautics ; Model Experiments.—Equations similar to (4·01) are capable of considerable use in aeronautics, where the forces and couples experienced by an airship and the various parts of aeroplanes when in motion through the air are required. In some aeronautical cases, and when the resistance of projectiles is concerned, the effect of the compressibility of the fluid (considered in Chapter V) has to be taken into account.

In order to determine $f(ur\rho/\mu)$ of equation (4·01), for different values of $ur\rho/\mu$, for any body, experiments may be made using the actual body ; or experiments with a geometrically similar model may be more convenient. The values of $ur\rho/\mu$ that occur when the full scale body is in motion, should be used in the model experiments. If the linear dimensions of the model be $1/n$ of those of the full scale body, and if the model be placed in the same fluid as occurs in the full scale case, the velocities used in the model experiments must be n

* E. G. Richardson, *Phys. Soc. of Lond.* (read March 13th, 1925).

† *Tech. Report of Advisory Committee for Aeronautics*, 1911–12, p. 96

times those occurring in the full scale. Thus very high speeds are required for a small model, and these may be difficult to attain. Also the velocity may, in the case of a gas, approach that of sound in the gas. Under these circumstances compressibility effects would occur in the model experiments, though they might be negligible in the full scale case.

Two methods may be used to avoid the difficulty introduced by these high velocities. Firstly, if the model experiments show that $f(u\tau\rho/\mu)$ of equation (4·01) has almost a constant value at high values of $u\tau\rho/\mu$, it may be assumed that this value is approximately true for the larger values of $u\tau\rho/\mu$ that occur in the full scale case. Secondly, by using in the model experiments a fluid that has a small value of μ/ρ, the values of u required may be reduced. The quantity μ/ρ is termed the *kinematic viscosity*, and may be denoted by the symbol ν. The effect of change of fluid may be estimated by means of the following table :—

TABLE II

(15° C. and 76 cm. of mercury.)

	μ	ρ	$\nu = \mu/\rho$
Glycerine	13·5	1·26	10·7
Hydrogen	0·000,089	0·000,085	1·05
Air	0·000,177	0·001,22	0·145
Water	0·011,5	1·00	0·011,5
Mercury	0·015,9	13·6	0·001,17

It will be seen that by substituting water for air in the model experiments the kinematic viscosity is reduced to about 1/12·6 of its former value, and a corresponding reduction in u may be made. Also no appreciable compressibility effects have to be considered when water is used. Care must, however, be taken that the pressure does not become so low at any part of the system that the dissolved gases in the water are evolved, or the water boils. This effect may occur near the blades of a propeller, and is termed 'cavitation.'

Several methods of experimenting with models* have been used for obtaining information about a body moving relative to the surrounding fluid. The body may be mounted at the end of a whirling arm in air or liquid, and the forces exerted on the body measured. It may be placed in a water channel through which water is passed, the stream lines and velocities being obtained by observing the motion of small particles placed in the liquid, the total force acting on the body and the pressure at different parts of the body's surface being measured. Corresponding experiments may be made in a wind channel. And, finally, in the case of certain bodies, the resistance to motion may be found by measuring their rate of fall through liquid or gas.

Certain difficulties are common to most of these methods. It is necessary that the model shall be geometrically similar to the full scale body; and this similarity should extend to the surface roughness as well as to the general contour of the body. The question arises as to whether the small initial turbulence of the fluid, before it approaches the body, is similar to that that will occur in the full scale case. This difficulty cannot be fully surmounted unless the exact state of the air in the full scale case be known. In all except the falling body experiments, the body must be supported (by wires, rods, etc.); and the disturbance caused by the supports has to be allowed for. If the effect be small, the total force exerted may be considered equal to the sum of those exerted on the body alone and the supports alone. But this is only an approximate means of correction. Distortion of the model may also occur; and it is not generally possible to arrange that this effect is geometrically similar to that occurring with the full scale body.

When experimenting with a rotating arm, a general circulation of the fluid is set up, and this may be corrected for as in

* *Dict. of Applied Physics*, Vol. v, p. 191.

the work on Pitot tubes (note, p. 23). In channel experiments it is found that the velocity may be measured by observing the motion of individual particles, that are of almost the same density as the fluid, or are very small, so that their natural rate of fall is small. Observations of a cloud of particles will give inaccurate estimates of the velocity. The motion in any channel will become turbulent if the critical speed be exceeded; and also the velocity in any case is not constant over a transverse section. In a wind channel, the motion of air is generally produced by means of a fan near the exit of the channel. Care has to be taken to keep the speed of the fan constant, and to design the fan so that periodic pulsations of velocity are only small. A grid of honeycomb form is placed at each end of the experimental portion of the channel, in order to make the velocity as nearly uniform and parallel to the walls as possible, over the central parts of the channel. Even when the velocity varies very little over the transverse section, it may be no longer constant when the body is introduced. When the diameter of the body is comparable with that of the channel, corrections have to be applied to reduce the observations to the case where the body is at a large distance from all obstructions. The total force acting on the body in three mutually perpendicular directions, and the three corresponding turning couples may be determined by means of a suitable balance. And the pressure at any point on the surface of the model may be measured by using a small hole in the side of a fine tube that is let into a groove in the surface of the model. In the case of a falling body, the total resistance to motion is the only quantity that can easily be measured; but rotation of the body may occur, thus rendering the results inaccurate.

If a Pitot tube be used to estimate the velocity near a model, care must be taken that the correct static pressure is observed. If the static pressure varies rapidly from point to point, it

may be necessary to insert the Pitot and static tubes in turn, at the same point. The direction of flow must also be known before the tubes can be correctly placed. In order to surmount this difficulty, four similar Pitot tubes may be used, symmetrically placed, so that their axes all meet at a point a short distance in front of the tubes. When this instrument is orientated so that the four observed pressures are equal, the axis of the instrument is in the direction of the velocity at the place. If, instead of employing a Pitot tube, the pressure be measured at the surface of a sphere or cylinder, on the side away from the arriving fluid, it is not necessary to know the direction of motion accurately, as the pressure is approximately constant over a considerable area (see Fig. 36). But there is the disadvantage in this method that the pressure reduction observed is not accurately proportional to the square of the velocity. In place of any of these methods, a hot wire anemometer may be used to determine the velocity.

Motion of a Body Floating on a Liquid ; Ship Models.—When a body floating on a liquid is moving relative to the liquid, work is done in producing eddies near the surface of the body. Work is also done in giving kinetic and potential energy to the waves produced on the surface of the liquid. Thus the force F required to maintain a constant relative velocity u, depends now on one more variable g. By the method of dimensions the form of the relationship is thus found to be

$$\frac{\mathrm{F}}{u^2r^2\rho}=f\left(\frac{ur\rho}{\mu},\frac{gr}{u^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (4{\cdot}04)$$

The function f cannot in general be kept unchanged, unless both $ur\rho/\mu$ and gr/u^2 be kept unchanged. If the linear dimensions r are changed, but the same liquid is used, it is not possible to satisfy these conditions, for both ur and r/u^2 would have to remain unchanged in value. However,

by supposing the force F to be equal to the sum of two approximately independent forces, due to 'skin friction' and 'wave production' respectively, we may write

$$F = F_1 + F_2 = u^2r^2\rho . f_1\left(\frac{ur\rho}{\mu}\right) + u^2r^2\rho . f_2\left(\frac{gr}{u^2}\right) \quad (4{\cdot}05)$$

The force F_1 due to skin friction may be considered approximately as a force per unit area of the 'wetted surface,' proportional to the square of the speed u. This assumes f_1 constant, and approximately equal to the value obtained by experimenting on long thin laths, towed through water in the direction of their length. (This term is in ship model experiments often assumed proportional to r^2 and $u^{1{\cdot}85}$; but this can only be considered an empirical assumption to give the most accurate results, possibly counterbalancing errors introduced by assuming F_1 and F_2 independent.) If the value of r/u^2 be kept unchanged, it will be seen that F_2 is proportional to $r^3\rho$.

This method of treatment (developed first by W. Froude)* is used to estimate the force that would be required to propel a ship at a chosen speed,† from measurements of the force required to propel a geometrically similar ship model. The model is experimented on whilst its speed is such that r/u^2 is the same for the model as that proposed for the ship. From the observed force exerted on the model, the part due to skin friction is deducted (being calculated approximately from previous experiments on towed laths).‡ The difference represents the force due to the wave production of the model. The corresponding force due to the waves produced by the proposed ship is calculated from this, as it is proportional

* *Inst. Naval Architects Trans.*, Vols. xv and xxiv. *Brit. Association Report*, 1872.

† *Dict. of Applied Physics*, Vol. i, p. 711.

‡ Stanton and Marshall, *Inst. Naval Architects Trans.*, 66, p. 214, 1924, and *Engineering*, 117, p. 718, May 30, 1924.

to the cube of the linear dimensions. Finally, to this is added the predicted 'skin friction' for the ship (again calculated from previous experiments on the towing of laths, etc.). The sum thus found is the predicted total force required to propel the ship at the constant specified speed. An allowance may have to be made if the model is used in fresh water, and the ship in the sea. When the value of r/u^2 is kept unchanged, the wave system produced by the moving body is of constant form, apart from any effects of surface tension and viscosity.

CHAPTER V

THE EFFECT OF COMPRESSIBILITY OF THE FLUID

Volume Elasticity.—When a fluid of appreciable compressibility is concerned, the density can no longer be regarded as constant. Work may be done in compressing the fluid as it moves through the system. The volume elasticity, defined as $k = -v\delta p/\delta v$ (where δp and δv are corresponding increases in pressure and volume) is dependent on whether the heat produced by the compression is supposed conducted away or not. The simplest cases are the isothermal (where the process takes place slowly and the temperature remains sensibly the same as the surroundings), and the adiabatic (where the change in volume takes place fast and the heat produced is not conducted away). In the case of liquids, these two compressibilities only differ by a few per cent. But in the case of a gas it is necessary to know to what extent conduction occurs. In most cases when the compressibility of the gas is important, the rate of flow is great; and hence conduction is small, and it may be assumed that the change is approximately adiabatic. The volume elasticity is then equal to γp, where γ is the ratio of the principal specific heats of the gas. Here two extra variables are introduced, γ and the absolute pressure p. If desired, the velocity of sound in the gas, $c = \sqrt{\dfrac{\gamma p}{\rho}}$ may be used in place of one of the above

extra variables. Any heat produced internally owing to the action of viscosity, will not cause the immediate need of any fresh variables, provided conduction is not important. But, as the pressure and density of the gas vary along the system, the temperature will vary, and the viscosity is therefore no longer a definite constant throughout the system. However, near any point the variables concerned are dp/dx, u, r, ρ, μ, γ and p.

Compressional or Sound Waves.—Let us consider a plane compressional wave in the fluid, the effect of viscosity being neglected. Let the wave front be normal to OX, and let particles originally at the plane x be in the plane $x + \xi$ at some chosen time. Those originally at $x + \delta x$ will be at

$$x + \delta x + \left\{ \xi + \frac{d\xi}{dx}\delta x \right\}.$$

The volume of unit mass of the fluid has become $v = v^0 + \delta v = v_0\left(1 + \frac{d\xi}{dx}\right)$, and the pressure has become $p = p_0 + \delta p = p_0 - k\frac{dv}{v} = p_0 - k\frac{d\xi}{dx}$ (where v_0 and p_0 are the specific volume and the pressure when the particles are in their original undisturbed positions).

Considering next a cylinder of the fluid, the length of which extends from x to $x + \triangle x$, and the base of which is of unit area. The resultant force acting on it in the direction OX may be put equal to the mass times the acceleration produced. Thus

$$p - \left(p + \frac{dp}{dx}\triangle x\right) = \left(\rho . \triangle x\right)\frac{d^2(x + \xi)}{dt^2}$$

or

$$\rho\frac{d^2\xi}{dt^2} = -\frac{dp}{dx} = k\frac{d^2\xi}{dx^2}$$

The solution of the equation

$$\frac{k}{\rho}\frac{d^2\xi}{dx^2}=\frac{d^2\xi}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}01)$$

is $\xi = f_1\left(\frac{t}{\tau} - \frac{x}{\lambda}\right) + f_2\left(\frac{t}{\tau} + \frac{x}{\lambda}\right)$, where $c = \frac{\lambda}{\tau} = \sqrt{\frac{k}{\rho}}$

which represents two waves, moving in the directions OX and XO, with velocity c.

In the case of a liquid, compressional waves chiefly occur in phenomena such as the 'water hammer,'* produced by suddenly stopping the liquid flow in a tube. The problem is complicated by the elasticity of the walls (of the extent to which they 'yield'), and it will not be considered further here. In the case of a gas, the conditions being supposed adiabatic, we have $k = \gamma p$ and

$$c = \sqrt{\frac{\gamma p}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}02)$$

The Equation of Continuity and Bernoulli's Equation.—For a gas, considering a tube of flow (Fig. 3) we have

$$m = a_1u_1/v_1 = a_2u_2/v_2 \quad . \quad . \quad . \quad (5{\cdot}03)$$

the equation of continuity (where v_1, v_2 are specific volumes).

Next let us derive Bernoulli's equation for a gas (see equation 1·02). The work done by the pressure exerted by the gas in the adjacent parts of the tube of flow, when unit mass passes into and out of the section, is

$$p_1v_1 - p_2v_2, \text{ or } - d(pv), \text{ or } - (p.dv + v.dp).$$

Energy is also supplied by the expansion of the gas, the amount when unit mass has passed being equal to $\int p.dv$ or $p.dv$.

* Gibson, *Water Hammer in Hydraulic Pipe Lines* (1908); also Constantinesco, *The Theory of Sonics* (1920).

The corresponding gain of potential energy of the gas is

$$(h_2 - h_1)g \text{ or } g.dh$$

and the gain of kinetic energy

$$\tfrac{1}{2}(u_2^2 - u_1^2) \text{ or } \tfrac{1}{2}d(u^2).$$

Hence, equating the energy supplied and the gain in energy we have

$$-(p.dv + v.dp) + p.dv = g.dh + \tfrac{1}{2}d(u^2)$$

$$\left.\begin{aligned} \text{or} \quad & \tfrac{1}{2}d(u^2) + g.dh + v.dp = 0 \\ \text{or} \quad & \tfrac{1}{2}[u^2] + g[h] + \int v.dp = 0 \end{aligned}\right\} \quad . \quad . \quad (5{\cdot}04)$$

when taken between two points along the stream line. (When v or $\dfrac{1}{\rho}$ is constant, this reduces to equation 1·02.)

Flow of Gas through Nozzles and Orifices.—This equation may be applied to the flow of a gas through a con-

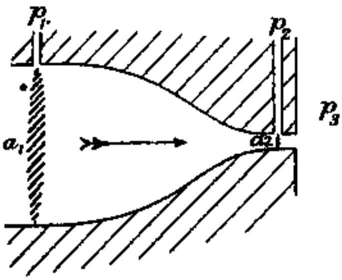


FIG. 39.

verging nozzle, or cone (or the entrance cone of a Venturi tube) (Fig. 39). If $pv^\gamma = K$, we have $\int v.dp = \int K^{1/\gamma}p^{-1/\gamma}.dp = \dfrac{\gamma K^{1/\gamma}}{\gamma - 1}\left[p^{1-\frac{1}{\gamma}}\right]$. Thus equation (5·04) becomes:

$$\frac{\gamma}{\gamma-1}(p_1^{1/\gamma}v_1)\left\{p_2^{\frac{\gamma-1}{\gamma}}-p_1^{\frac{\gamma-1}{\gamma}}\right\}+g(h_2-h_1)+\tfrac{1}{2}(u_2^2-u_1^2)=0$$

The term $g(h_2 - h_1)$ due to differences in level, can generally be neglected; though it should be noted that the reading of a differential manometer does not exactly eliminate this term. For, if the manometer were lowered a certain distance bodily, the equal columns of gas introduced in the two connecting tubes have different densities, and a slightly different reading would result.

Omitting this term, and using equation 5·03 we obtain

$$\frac{\gamma}{\gamma-1}p_1^{1/\gamma}v_1\left\{p_1^{\frac{\gamma-1}{\gamma}}-p_2^{\frac{\gamma-1}{\gamma}}\right\}=\tfrac{1}{2}m^2v_1^2\left\{\left(\frac{v_2/v_1}{a_2}\right)^2-\frac{1}{a_1^2}\right\} \quad (5·05)$$

Now $v_2/v_1 = (p_1/p_2)^{1/\gamma}$ and hence

$$m = a_2\sqrt{\frac{\frac{\gamma}{\gamma-1}p_1\rho_1\left\{1-(p_2/p_1)^{\frac{\gamma-1}{\gamma}}\right\}(p_2/p_1)^{2/\gamma}}{1-\left(\frac{a_2}{a_1}\right)^2\left(\frac{p_2}{p_1}\right)^{2/\gamma}}} \quad (5·06)$$

(When $p_1 - p_2$ is small compared to unity this equation reduces to equation (2·04), that being applicable when the density is sensibly constant.) The temperature, at any point where the pressure is known, may be deduced; since $\theta p^{\frac{1-\gamma}{\gamma}}$ is a constant, if no heat be produced due to viscosity, and if no conduction of heat occurs.

From the above equation it would appear that, if p_1 and ρ_1 were kept constant, and p_2 gradually decreased, then the value of m would increase till it reached a maximum value and subsequently would decrease towards zero. In experiments it is found that when p_3 is decreased, this maximum value is reached, but no subsequent decrease is found, the values of m and p_2 remaining unchanged as p_3 is decreased further. (The decrease of m, not found experimentally, may be con-

sidered to correspond to what would happen if the tube converged from a_1 and diverged again to reach a_2. The form of the conical tube between a_1 and a_2 has not been made use of in the above calculations, and thus the equation would apply to either form of tube.)

From equation (5·06) it appears that when m has reached its maximum value, the velocity at a_2 is equal to that of a compressional or sound wave in the gas at that point. Let us work out only the case when (a_2/a_1) is small and can be neglected in the equation. The value of p_2/p_1 to make

$$\left\{1-(p_2/p_1)^{\frac{\gamma-1}{\gamma}}\right\}(p_2/p_1)^{2/\gamma}$$

a maximum, is required. By differentiating with respect to (p_2/p_1) treated as a single variable, and equating to zero, we get

$$\frac{p_2}{p_1}=\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \quad . \quad . \quad . \quad . \quad (5{\cdot}07)$$

for m to be a maximum. Substituting in equation (5·06) we find

$$m_{(maximum)}=a_2\sqrt{\gamma p_2\rho_2}=a_2\rho_2c_2 \quad . \quad . \quad (5{\cdot}08)$$

where c_2 is the velocity of sound at the point. Hence, for the mass per second to reach its maximum, the velocity u_2 must be equal to that of sound. If the velocity at a_2 could become equal to, or greater than, that of sound at the point, any small change in the pressure p_2 at points beyond the section a_2 could have no effect on the velocity at a_2. For the change in pressure produces an effect at a distant point by a wave of compression or rarefaction travelling to that point; and such a wave could not travel in the opposite direction to the direction of flow, if the velocity of flow were itself equal to or greater than the wave or sound velocity. Hence, the pressure p_2 beyond a_2 has no effect on the velocity

at a_2, when p_3 is less than the value of p_2 given by equation (5·07). For such a value of p_3, p_2 retains the (minimum) value given by equation (5·07) and the mass per second retains the (maximum) value given by equation (5·08).

Another way of arriving at the same conclusion is to write $p_2 - p_1 = \delta p$, $a_2 - a_1 = \delta a$ in equation (5·06) neglecting quantities of the second order of smallness. Thus we obtain

$$\left(\frac{m}{a}\right)^2 = \frac{\rho . \delta p}{\dfrac{\delta a}{a} + \dfrac{1}{\gamma}\dfrac{\delta p}{p}}$$

or

$$\frac{r}{u^2\rho} \cdot \frac{dp}{dx} = \frac{\dfrac{r}{a}\dfrac{da}{dx}}{1 - u^2/c^2} \quad . \quad . \quad . \quad . \quad (5·09)$$

where r is the radius of the tube, and x is measured parallel to the axis of the tube in the direction of flow. Thus, if at any point in the tube $u = c$, there is an infinite pressure gradient there, unless the tube is parallel-sided at the point.

Thus, when the pressure at the exit of the system is gradually decreased, the velocity increases at all points, till the greatest velocity (u_2 at the narrowest part of the tube) reaches that of sound in the gas at the point. Further decrease in the pressure at the exit leaves the conditions unchanged at all points between the entrance and the narrowest section (a_2). Experiments using converging nozzles and Venturi tubes are in fairly close agreement with this theory. Any deviation between the theory and experiment is probably due to the neglect of viscosity and conduction in the calculations.

If in place of a nozzle, an orifice* be used, the narrow section concerned is the 'vena contracta.' As the ratio of the pressures on the two sides becomes large, the vena

* Hodgson, *Proc. Inst. Civ. Engineers*, Vol. cciv, p. 131, 1918.

contracta becomes bigger and finally almost coincides with the orifice. Thus the mass per second does not reach a maximum for a ratio of pressures equal to that given by equation (5·07). In the case of nozzles,* the value of m increases to a certain extent, even after the value of p_2 has become less than that of p_2 calculated from equation (5·07); for the action of viscosity was not considered in the calculation that predicted a definite maximum value of m, given by equation (5·08).

In the case either of a converging nozzle or of an orifice, the emergent jet† is found to become striated when the velocity at the narrow section has about reached that of sound at the point. The striations (which are stationary sound waves) may be rendered visible by suitable lighting and by the presence of some water vapour in the gas passing through. As p_2 is decreased the striations move further apart. By placing along the axis of the jet a small rod, in the side of which a static pressure hole (Fig. 6) is drilled, the pressure at different points along the axis may be measured. When striations exist, corresponding periodic variations of pressure with distance along the jet are found to occur. The pressure does not simply decrease steadily along the jet in the direction of motion, but has the periodic variations superposed on the general decrease in pressure. The pressure in the jet may at points be less than p_2, the pressure of the surrounding gas.

The Pitot Tube.—If a Pitot tube be used at a place where the velocity of the gas is less than, but comparable with, that of sound in the gas at the point, the theory may be developed in a similar manner to that yielding equation (4·06). Thus from equation (5·05), where suffix unity now denotes a point in the fluid in front of the apparatus, and suffix two

* Hartshorn, *Proc. Roy. Soc.*, 94A, p. 155, 1918.

† Rayleigh, *Phil. Mag.*, p. 177, 1916.

denotes a point inside the Pitot tube where $u_2 = 0$, we have

$$u_1^2 = \frac{2\gamma}{\gamma - 1} \cdot \frac{p_1^{1/\gamma}}{\rho_1} \left\{ p_2^{\frac{\gamma-1}{\gamma}} - p_1^{\frac{\gamma-1}{\gamma}} \right\} \quad . \quad (5\cdot10)$$

When u_1 exceeds the velocity of sound c_1, the equation becomes different,* but will not be derived here. Such a case would occur at the tip of a projectile moving at a speed greater than that of sound in the air.

Motion of a Body through the Fluid.—The force F required to keep a body moving through a compressible fluid at a constant velocity relative to the fluid, is seen by the method of dimensions to be of the form

$$\frac{F}{u^2 r^2 \rho} = f\{ur\rho/\mu,\ \gamma,\ u^2\rho/\gamma p\} \quad . \quad . \quad . \quad (5\cdot11)$$

If only diatomic gases, having $\gamma = 1\cdot4$ approximately, are concerned, the function involves two non-dimensional products. Thus a series of curves (or a surface) is required to relate these two products and $\frac{F}{u^2 r^2 \rho}$. This mode of treatment has been used in the study of projectiles. The curves in general exhibit a peculiarity where the velocity equals that of sound. (This treatment assumes that the change of μ with temperature, from point to point in the system, may be neglected.)

The Conservation of Energy.—Finally, let us consider the conservation of energy as applied to flow of a perfect gas,† any heat produced by the action of viscosity increasing the temperature and internal energy of the gas. Then

$$-d(pv) = d(u^2/2) + g.dh + dU \quad . \quad . \quad (5\cdot12)$$

where dU is the change in internal energy of unit mass of the

* *The Mechanical Properties of Fluids*, p. 346, 1923 (Blackie).

† Zeuner, *Technical Thermodynamics*, 1907. Goodenough, *Principles of Thermodynamics*, 1912.

gas. The value of U for a perfect gas may be found by calculating the work done when unit mass of the gas expands adiabatically to an infinite volume, no work being done in increasing the distance between the molecules. Let $pv^{\gamma} = K$.

Then

$$U = \int_{v=v}^{v=\infty} p.dv = \frac{K}{1-\gamma}\Big[v^{1-\gamma}\Big]_{v=v}^{v=\infty} = \frac{pv}{\gamma - 1}$$

where p and v are the original pressure and specific volume. Hence equation (5·12) becomes

$$d(u^2/2) + g.dh + \frac{\gamma}{\gamma - 1}d(pv) = 0 \quad . \quad . \quad (5\cdot13)$$

along any tube of flow.

This equation is true for any motion of the gas, provided no heat is conducted across the walls of the system, and provided the forces acting between molecules when at a distance from one another are neglected. (Equation 5·04 is only true when the action of viscosity can also be neglected.)

If the gas pass through a porous plug, or other constriction, there is thus no difference in pv, or the temperature θ, between two points at the same level, at both of which u is sensibly zero. In the Joule-Thomson porous plug experiment* however, small differences in temperature were found, due to the gas not obeying the perfect gas laws (i.e. due to the forces between the molecules).

Equation (5·13) may be applied to the flow of a gas in straight stream-lines of constant cross section. We then obtain

$$v.dp + \left\{1 + (\gamma - 1)\frac{u^2}{c^2}\right\}p.dv = 0 \quad . \quad (5\cdot14)$$

* *Phil. Mag.*, 4th series, Vol. iv, 1852; *Phil. Trans.*, 1853, 1854 and 1862.

where $c = \sqrt{\frac{\gamma p}{\rho}}$, the velocity of a compressional or sound wave in the gas at the point. If the gas be supposed just near a chosen point to follow a law of expansion $pv^n =$ constant, we find therefore that

$$n = 1 + (\gamma - 1)\frac{u^2}{c^2} \quad . \quad . \quad . \quad . \quad (5{\cdot}15)$$

We see that if u be small, $n = 1$ and the flow is iso-thermal; and if the velocity is ever to be equal to that of sound, we must have $n = \gamma$, the flow adiabatic, and no heat produced by viscous forces. (In other words, if heat be produced by viscous forces, an infinite pressure gradient would be required to make the velocity equal to that of sound.* Previously it was shown by equation (5·09) that in a tube that was not parallel sided, an infinite pressure gradient was required, even when no viscous forces were acting.)

nd, *Proc. Phys. Soc.*, Vol. xxxvi, p. 367, Aug. 1924.

NUMERICAL DATA

Viscosities.—See Table I, p. 11.

Densities and Kinematic Viscosities.—See Table II, p. 72.

Viscosities of Mixtures of Glycerine and Water:

TABLE III.

Percentage of Glycerine	100	80	60	40	20	0
Density	1·26	1·21	1·15	1·10	1·05	1·00
Viscosity (at 18° C.) .	10·0	0·56	0·11	0·040	0·019	0·0105

(The above approximate values are useful in obtaining mixtures for laboratory experiments, having suitable values of μ/ρ.)

Surface Tensions:

TABLE IV

Water	74	Carbon Bisulphide	34
Mercury	527	Glycerine	63
Water—Mercury	427		

(The above are for use in approximate calculations. The values vary with temperature and any slight contamination of the surface.)

Ratio of Specific Heats of Gases ($\gamma = c_p/c_v$):

TABLE V

Monatomic gases, Argon, Helium	1·66
Diatomic gases, Hydrogen, Oxygen, Nitrogen, Air . .	$1{\cdot}40_4$
Carbon dioxide (tri-atomic)	1·30

The Mass per Second (m) of Fluid Passing through a Tube may be estimated by means of a Venturi tube (Fig. 22), or an orifice consisting of a simple circular hole drilled in a thin plate (Fig. 13). (Diameter of hole at least twenty times the thickness of the plate.) The following formulæ may be used:

Liquid flow, or slow gas flow ($p_2/p_1 > 0{\cdot}98$).

$$m = Ca_2\sqrt{\frac{2\rho p_{12}}{1 - (a_2/a_1)^2}}$$

C (Venturi) = 0·97 approximately. (Unless a very small diameter tube.)

C (Orifice) = 0·608 (if A and B, Fig. 13, are close to the surfaces of the plate, and if $a_2/a_1 < 0{\cdot}5$). (For the effect of viscosity, see Fig. 23.)

Rapid gas flow ($p_2/p_1 \not< 0{\cdot}98$).

Equation (5·06) may be used, with a multiplying factor C:

C (Venturi) = 0·97 approximately (except for small tubes and very rapid rates of flow).

C (Orifice) = $0{\cdot}914 - 0{\cdot}306\ p_2/p_1$ (A and B, Fig. 13, close to plate, and a_2/a_1 small). (When p_2/p_1 lies between 1 and 0·9 in value the simpler liquid flow equation given above may be used for an orifice, ρ denoting the density of the gas before reaching the orifice.)

(For other constants and methods of evaluating equation (5·06), see Hodgson, *Proc. Inst. Civ. Engineers*, Vol. cciv, p. 108, 1918.)

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